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Bi-partite entanglement entropy in massive (1+1)-dimensional quantum field theories

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Abstract

This paper is a review of the main results obtained in a series of papers involving the present authors and their collaborator J L Cardy over the last 2 years. In our work, we have developed and applied a new approach for the computation of the bi-partite entanglement entropy in massive (1+1)-dimensional quantum field theories. In most of our work we have also considered these theories to be integrable. Our approach combines two main ingredients: the 'replica trick' and form factors for integrable models and more generally for massive quantum field theory. Our basic idea for combining fruitfully these two ingredients is that of the branch-point twist field. By the replica trick, we obtained an alternative way of expressing the entanglement entropy as a function of the correlation functions of branch-point twist fields. On the other hand, a generalization of the form factor program has allowed us to study, and in integrable cases to obtain exact expressions for, form factors of such twist fields. By the usual decomposition of correlation functions in an infinite series involving form factors, we obtained exact results for the infrared behaviours of the bi-partite entanglement entropy, and studied both its infrared and ultraviolet behaviours for different kinds of models: with and without boundaries and backscattering, at and out of integrability.

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(Some figures in this article are in colour only in the electronic version)

Contents

1. Introduction

Entanglement is a fundamental property of quantum systems. Its most striking consequence is the fact that local measurements may display stronger correlations than would be expected classically—in quantum mechanics, performing a local measurement may affect instantaneously the state of a quantum observable far away. This is in fact one of the main differences between quantum and classical systems. Given that our understanding of the physical world is largely based on everyday experience and that entanglement seems to contradict such experiences, it is not surprising that the existence of quantum entanglement has been a source of controversy and heated scientific debate for some time (see e.g. [1] for the famous EPR-paradox). Today entanglement is a well-established and measurable phenomenon, whose reality was experimentally confirmed in the early 1980s by the famous experiments of Alain Aspect and collaborators [2, 3], using pairs of maximally entangled photons. In the last decades, many applications of entanglement have developed into successful fields of research such as quantum computation and quantum cryptography. Entanglement lies also at the heart of other interesting phenomena such as quantum teleportation (see [4] for a review).

As a consequence of the prominent role of entanglement in quantum physics, there has been great interest in developing efficient (theoretical) measures of entanglement. For example, a quantity of current interest in quantum models with many local degrees of freedom is the bi-partite entanglement entropy [5], which we will consider in this paper. In its most general understanding, it is a measure of the amount of quantum entanglement, in some pure quantum state, between the degrees of freedom associated with two sets of independent observables whose union is complete on the Hilbert space. In the cases considered in this paper, the quantum state will mostly be the ground state $|\mathrm{gs}\rangle$ of some extended (1+1)-dimensional (1 space + 1 time dimension) quantum model, and the two sets of observables correspond to the local observables in two connected regions, say A and its complement, $\bar{A}$ (we will also briefly discuss the general technique in the case of excited states). Other measures of entanglement exist, see e.g. [5–9], which occur in the context of quantum computing, for instance. Measures of entanglement are important at a theoretical level, as they give a good description of the

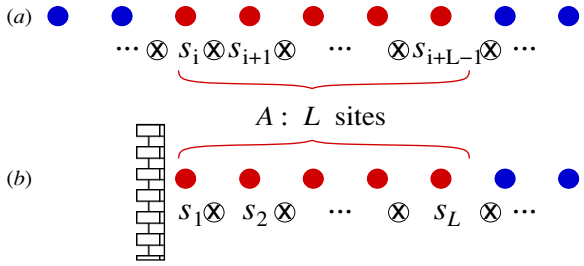

Figure 1. The entangled region A of length L of a quantum spin chain. (a) The bulk case and (b) the boundary case.

quantum nature of a ground state, perhaps more so than correlation functions. However, in contrast to the theoretical understanding of entanglement measures, the development of quantitative experimental measurement methods of entanglement is still in its infancy, see a proposition using quantum noise in [11].

General aspects of the entanglement entropy in extended quantum systems will be discussed in [10]. For our present purposes, prominent examples of extended one-dimensional quantum systems are quantum spin chains, which model physical systems consisting of infinitely long one-dimensional arrays of equidistant atoms characterized by their spin. Their entanglement has been extensively studied in literature [12–20].

In order to provide a formal definition of the entanglement entropy, let us consider the Hilbert space of a quantum model, such as the chain above, as a tensor product of local Hilbert spaces associated with its sites. This can be written as a tensor product of the two Hilbert spaces associated with the regions A and $\bar{A}$:

$$\mathcal{H} = \mathcal{A} \otimes \bar{\mathcal{A}}. \tag{1.1}$$

Then the entanglement entropy is the von Neumann entropy of the reduced density matrix ρ_A associated with A:

$$S_A = -\mathrm{Tr}_{\mathcal{A}} \rho_A \log \rho_A, \qquad \rho_A = \mathrm{Tr}_{\bar{\mathcal{A}}} |\mathrm{gs}\rangle\langle \mathrm{gs}|. \tag{1.2}$$

We will be interested in analysing the entanglement entropy in the scaling limit of infinite-length quantum chains. The scaling limit gives the universal part of the quantum chain behaviour near a quantum critical point, which is described by a model of (1+1)-dimensional quantum field theory (QFT) (which we will assume throughout to possess Poincaré invariance). The scaling limit is obtained by approaching the critical point while allowing the length L of the region A go to infinity in a fixed proportion with the correlation length ξ (these lengths are measured in number of lattice sites). In this limit, the entanglement entropy is in fact divergent. The way the entanglement entropy diverges has been understood in [22–24]. It is controlled by the central charge c corresponding to the critical point that is being approached. In general, for every point that separates a connected component of A from a connected component of its complement $\bar{A}$ (a boundary point of A), there is a term $c/6 \log \xi$. We will look at two cases: where the quantum chain is infinite in both directions and A is a connected segment, with two boundary points (the bulk case), and where the quantum chain is infinite only in one direction, and A is a connected segment starting at the boundary of the chain, with only one true boundary point (the boundary case)—see figure 1. In these two cases, the divergent part of the entanglement entropy is respectively:

$$S_A^{\mathrm{bulk}} \sim \frac{c}{3} \log \xi + O(1), \qquad S_A^{\mathrm{boundary}} \sim \frac{c}{6} \log \xi + O(1). \tag{1.3}$$

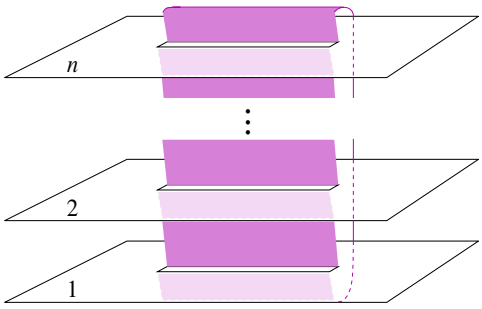

Figure 2. A representation of the Riemann surface $\mathcal{M}_{n,x_1,x_2}$.

These divergent terms are not universal (as they depend on the correlation length); the universal terms of the entanglement entropy are hidden in the $O(1)$ part. These universal terms, that depend on the proportion to ξ with which A is sent to infinity, are in general described by QFT (conformal, massive, etc). The analysis of the universal terms, in the exactly critical case using CFT techniques in various situations is reviewed in [25]. The general techniques of QFT were also used to study the universal evolution of the entanglement entropy in quench situations [26–28]. In the present review, we will explain how to use massive QFT techniques in order to obtain information about the universal terms near criticality. We will now overview the main ideas and results of our works (partly with J L Cardy) on this subject.

1.1. Overview of ideas and results

It is known since some time [22–24] that the bi-partite entanglement entropy in the scaling limit can be re-written in terms of more geometric quantities, using a method known as the 'replica trick'. The essence of the method is to 'replace' the original QFT model by a new model consisting of n copies (replicas) of the original one, in order to use the formula

$$S_A = -\lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n} \mathrm{Tr}_{\mathcal{A}} \rho_A^n. \tag{1.4}$$

The trace in this formula is reproduced by the condition that these copies be connected cyclicly through a finite cut on the region A. Then, this trace is the partition function $Z_n(x_1, x_2)$ of the original (euclidean) QFT model on a Riemann surface $\mathcal{M}_{n,x_1,x_2}$ with two branch points, at the points x_1 and x_2 in $\mathbb{R}^2$, and n sheets cyclicly connected (we will provide more explanations about this in the following section). Figure 2 shows a representation of such a Riemann surface. The positions of the branch points correspond to the end points of the region A in the scaling limit. This gives

$$S_A(|x_1 - x_2|) = -\lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n} \frac{Z_n(x_1, x_2)}{Z_1^n}. \tag{1.5}$$

Here, $|x_1 - x_2|$ is the euclidean distance between x_1 and x_2. This formula holds both for our bulk and boundary cases. In the boundary case, one of the branch points (say x_1) is on the boundary of the model, $x_1 = (0, 0)$. The positions x_1 and x_2 are dimensionful positions in the QFT model. They are naturally at zero imaginary time $x_1 = (x_1, 0), x_2 = (x_2, 0)$ (but this is not crucial because the euclidean QFT has rotation invariance), and their x-coordinates are related to the ratio between the dimensionless region length L and correlation length ξ of the quantum chain by $|x_1 - x_2| = L/(m\xi)$, where m is the QFT mass scale associated with ξ (whose only role here is to provide a dimension).

Naturally, this expression implies that we must analytically continue the quantity $Z_n(x_1, x_2)$ from $n \in \mathbb{N}$, where it is naturally associated with Riemann surfaces, to $n \in [1, \infty)$. The object $\mathrm{Tr}_A \rho_A^n$ certainly has a well-defined meaning for any n such that $\mathrm{Re}(n) > 0$. Indeed, ρ_A is Hermitian (and has non-negative eigenvalues summing to 1), so that $\mathrm{Tr}_A \rho_A^n$ is the sum of the nth powers of its eigenvalues (with multiplicities). Note that this is an analytic continuation from positive integers n to complex n that satisfies the requirements of Carlson's theorem [29], hence the unique one that does. The scaling limit of this object is what defines the proper analytic continuation of $Z_n(x_1, x_2)$. Finding the correct analytic continuation has been one of the major challenges encountered in our work. It is natural to assume, as it has been done before [22] and discussed in [30] in the present context, that the two branch points just become conical singularities with angle $2\pi n$, the rest of the space being flat.

As will be described later in more detail, in [31] we showed that there is a way of associating the branch points at x_1 and x_2 with local QFT fields: through *branch-point twist fields* $\mathcal{T}(x_1), \tilde{\mathcal{T}}(x_2)$. These twist fields are defined only in the replica model (not in the original model), and are associated with certain elements of the extra permutation symmetry present in the replica model. In terms of these fields we showed that, in the bulk case,

$$\frac{Z_n(x_1, x_2)}{Z_1^n} = \mathcal{Z}_n \varepsilon^{2d_n} \langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|0\rangle. \tag{1.6}$$

Here $\langle 0|\cdots|0\rangle$ denote correlation functions in the n-copy model; the state $|0\rangle$ is the vacuum state of the latter. The branch-point twist fields have the CFT normalization (which we will discuss later). The constant $\mathcal{Z}_n$, with $\mathcal{Z}_1 = 1$, is an n-dependent non-universal constant, ε is a short-distance cut-off which is scaled in such a way that $\mathrm{d}\mathcal{Z}_n/\mathrm{d}n = 0$ at $n = 1$, and, finally, d_n is the scaling dimension of the counter parts of the fields $\mathcal{T}, \tilde{\mathcal{T}}$ in the underlying n-copy conformal field theory,

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right), \tag{1.7}$$

which can be obtained by CFT arguments [26, 31] and where c is the central charge. The short-distance cut-off is related to the correlation length via $\varepsilon = a/(m\xi)$ for some dimensionless finite non-universal number a. This short-distance cut-off takes care of the infinite contributions to the partition functions $Z_n(x_1, x_2)$, compared to the nth power of Z_1, around the points x_1 and x_2 where a branch point lies. There is one contribution of ε_n^d for each point, corresponding to one contribution of $c/6 \log \xi$ to the entanglement entropy.

In our most recent work [32], we have generalized this understanding to the boundary case, where now the region A extends between the origin $x_1 = 0$, where the boundary of the model is located, and the x-coordinate x_2. With similar arguments we have shown that

$$\frac{Z_n(0, x_2)}{Z_1^n} = \mathcal{Z}_n \varepsilon^{d_n} \langle 0|\mathcal{T}(x_2)|B\rangle. \tag{1.8}$$

The state $|B\rangle$ is a boundary state, which depends on the particular model and boundary condition under consideration. In the context of integrable models, it was introduced in the seminal work of Ghoshal and Zamolodchikov [33]. Here, we take it with the normalization $\langle 0|B\rangle = 1$. The factors $\mathcal{Z}_n$ and ε are defined through similar conditions as in the bulk case.

In terms of the variables $r = |x_1 - x_2|, \varepsilon, m$, and using the CFT normalization of the branch-point twist fields as well as large-distance factorization of correlation functions, we can

re-write the logarithmic divergence formula (1.3) in a more precise fashion. The IR (large-mr) and UV (small-mr) leading behaviours of the entropy in the bulk are

$$S_A^{\text{bulk}}(r) = \begin{cases} \dfrac{c}{3}\log(r/\varepsilon) + o(1) & \varepsilon \ll r \ll m^{-1} \\ -\dfrac{c}{3}\log(\varepsilon m) + U^{\text{model}} + O((rm)^{-\infty}) & \varepsilon \ll m^{-1} \ll r. \end{cases} \tag{1.9}$$

The IR leading behaviour is interpreted as the saturation of the entanglement entropy at large distances. Note our use of the 'small-o' notation $o(1)$ in describing the UV behaviour, representing here a correction that tends to zero as $rm \to 0$. The term U^{model} is a model-dependent universal constant. In (1.9), the short-distance cut-off φ is of course non-universal, and it is in general hard to evaluate its exact relation to the correlation length. But the UV leading behaviour provides an unambiguous QFT definition for it, independently of what the correlation length of a particular quantum chain may be. Once this definition is taken, the terms that are added to $\frac{c}{3}\log(\varepsilon)$ are universal—these terms constitute the universal part of the entanglement entropy. For instance, the constant U^{model} is indeed a universal QFT quantity. It was computed exactly for the first time in the Ising universality class in our work [30]; the expression evaluates to

$$U^{\text{Ising}} = -0.131\,984\ldots \tag{1.10}$$

in [31] using QFT methods, reproducing results of [16] obtained on the lattice. This is in agreement with the exact lattice calculations of [16] in the XX spin chain model, whose entanglement entropy can be related to that of the Ising quantum chain [34]. It is also in agreement with the more general IR lattice results of [35] (see also [26] and [36]) combined with the UV lattice numerical values obtained in [14] (since U^{model} is universal, it combines the IR and UV asymptotics). Note that in [27], we have in fact a universal normalisation constant for the more general quantity $\text{Tr}_{\mathcal{A}}\rho_A^n$, related to the so-called Renyi entropy (IR lattice results for this can be found in [37]. Under certain assumptions about a slow dependence on n of the UV behaviour of this quantity, this can allow one to study the full spectrum of ρ_A, in the whole universal regime near criticality, following methods used at criticality in [38]. In the present review, we will not discuss how to explicitly evaluate U^{Ising} or the related normalisation constant for $\text{Tr}_{\mathcal{A}}\rho_A^n$.

In the boundary case, our definitions of the boundary state further imply that the UV and IR leading behaviours of the entanglement entropy are

$$S_A^{\text{boundary}}(r) = \begin{cases} \dfrac{c}{6}\log(2r/\varepsilon) + V(\kappa) + o(1) & \varepsilon \ll r \ll m^{-1} \\ -\dfrac{c}{6}\log(\varepsilon m) + \dfrac{U^{\text{model}}}{2} + O((rm)^{-\infty}) & \varepsilon \ll m^{-1} \ll r. \end{cases} \tag{1.11}$$

Again, ε is non-universal and hard to calculate. But since we know that the constant U^{model} is universal, the IR behaviour in (1.11) gives an unambiguous definition of ε. Then, the terms that are added to $-\frac{c}{6}\log(\varepsilon)$ are the universal part of the entanglement entropy, true QFT quantities. In particular, the UV behaviour in (1.11) provides a universal definition for $V(\kappa)$. Note that the leading asymptotic term at large distance $U^{\text{model}}/2$ in the boundary case is just a choice. Once this choice is made, the constant $V(\kappa)$ is universally fixed. In particular, our short-distance cut-off ε here is in general different in the bulk and boundary cases—it is related to the correlation length ξ in different ways. Here, κ is a parameter that represents the boundary condition.

For an extensive discussion of the entanglement entropy in systems with boundaries, see the contribution [39] in this issue.

A major focus of our work [30–32] has been the study of the ratios of partition functions (1.6) and (1.8) at large distances $r = |x_1 - x_2|$ (the infrared (IR) region) for (1+1)-dimensional integrable QFTs. Integrability means that in these models there is no particle production in any scattering process and that the scattering (S) matrix factorizes into products of two-particle S-matrices which can be calculated exactly (for reviews, see e.g. [40–44]). This is the factorized-scattering theory of integrable models. Since the scattering matrix and the particle spectrum fully encode the local definition of QFT, it is also possible to incorporate the presence of boundaries in an integrable model defined in the factorized-scattering way. The study of integrable QFTs with boundaries has attracted a lot of attention in the last two decades (see e.g. [33, 45–49]). In our work on the boundary case we have made extensive use of the results of Ghoshal and Zamolodchikov [33], particularly the explicit realization of the boundary state which they proposed.

Taking the known S-matrix of a model as input it is possible to compute the matrix elements of local operators (also called form factors). This is done by solving a set of consistency equations [50, 51], also known as the form factor bootstrap program for integrable QFTs. In [31], this program was used and generalized in order to compute (1.6) in the case of integrable models with diagonal scattering matrix (that is, without backscattering). In order to do this, the two-point function in (1.6) was expressed as a sum in terms of form factors of the twist fields involved (an expansion using a decomposition in energy–momentum eigenstates). This was then extended to models with backscattering, such as the sine-Gordon model [30], to integrable models with boundaries in [32], and some aspects were generalized to non-integrable models [52].

One of the most interesting results of our works [31, 30, 52] has been the identification of the next-to-leading order correction to the large-distance (large-r) behaviour of the entropy of all (unitary) massive two-dimensional theories, that is, the third term in the following large-r expansion:

$$S_A^{\text{bulk}}(r) = -\frac{c}{3}\log(\varepsilon m) + U^{\text{model}} - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O(\mathrm{e}^{-3rm}). \tag{1.12}$$

Here, m_α are the masses of the ℓ particles in the QFT model, with $m \equiv m_1 \leqslant m_\alpha$, and U^{model} is the model-dependent constant introduced above. The first two terms are the expected saturation of the entanglement entropy, but the interesting feature is the universal third term, where we see that the leading exponential corrections are independent of the scattering matrix, and only depend on the particle spectrum of the model. This is quite striking: for instance, a model of ℓ free particles of masses m_α will give the same leading exponential corrections as one with interacting particles of the same masses. The result (1.12) was first obtained using integrable QFT methods [30, 31], then, even more strikingly, it was understood to hold as well outside of integrabilty [52].

Infrared corrections were also studied for integrable models with boundary [32], in which case they are always model dependent, in particular through the reflection matrices off the boundary.

The results described so far have been extended further for the particular case of the Ising model. For this model, the particular form of all infrared corrections to the entropy with and without boundary has been obtained in [32]. In fact, for this model without boundary, like for other free-field QFT models, there is an alternative powerful way of studying the entanglement entropy, see the review [53].

This paper is organized as follows. In section 2, we review the relationship between partition functions on multi-sheeted Riemann surfaces, correlation functions of branch-point twist fields and the entanglement entropy. Employing these relationships, we provide

expressions for the bi-partite entanglement entropy of (1+1)-dimensional quantum field theories, both in the bulk and boundary cases. In section 3, we introduce the form factor program for branch-point twist fields. We explain how it can be employed to obtain the form factors for these twist fields in integrable models, and how these form factors can be checked for consistency against conformal field theory results. For the two-particle form factors we generalize this program also to (1+1)-dimensional non-integrable QFTs. We identify the general structure of the two particle form factors and, in integrable cases, suggest how higher particle form factors may be obtained from lower particle ones. For the Ising model, we give closed formulae for all higher particle form factors. In general, we recall how the form factors can be regarded as building blocks for correlation functions. The correlation functions can then be expressed as series where the leading contributions at large distances arise from the lower particle form factors. In section 4, we used these form factor series in order to analyse the entanglement entropy, both in the bulk and boundary cases. In the bulk case, our most important result is the universal expression (1.12) for the next-to-leading order correction to the entanglement entropy at long distances. Both for the bulk and boundary cases of the Ising model, we evaluate all higher order infrared corrections to the entanglement entropy. In the boundary case, we find the precise relationship between the ultraviolet leading behaviour of the entanglement entropy and the boundary entropy introduced by Affleck and Ludwig [54]. Finally, in section 5 we summarize our conclusions and outlook.

2. Replica trick and entanglement entropy

2.1. Partition functions on multi-sheeted Riemann surfaces

From the considerations in section 1, it is clear that for the study of the entanglement entropy in the scaling limit, we must study partition functions of (euclidean-signature) quantum field theory on multi-sheeted Riemann surfaces. We will come back to the precise relation in the following section, but for now, let us discuss such partition functions. In particular, we wish to introduce the concept of branch-point twist fields, following [31].

The partition function of a model of (1+1)-dimensional QFT, with local Lagrangian density $\mathcal{L}[\varphi](x)$, on a Riemann surface $\mathcal{R}$ is formally obtained by the path integral

$$Z[\mathcal{L},\mathcal{R}] = \int [\mathrm{d}\varphi]_{\mathcal{R}} \exp\left[-\int_{\mathcal{R}} \mathrm{d}^2 x_{\mathcal{R}}\, \mathcal{L}[\varphi](x_{\mathcal{R}})\right]. \tag{2.1}$$

Here, $[\mathrm{d}\varphi]_{\mathcal{R}}$ is an infinite measure on the set of configurations of some field φ living on the Riemann surface $\mathcal{R}$ and on which the Lagrangian density depends in a local way, and $x_{\mathcal{R}}$ is a point on the Riemann surface. Consider Riemann surfaces with zero curvature everywhere except at a finite number of points; the points where the curvature is non-zero are branch points. In the case where the initial quantum system we consider is on the line, the Riemann surface forms a multiple covering of $\mathbb{R}^2$; in the case where the model is on the half-line, it forms a multiple covering of the half-plane, which we will take to be the right half-plane, $\mathbb{R}^2_{\rightarrow} \equiv \{(x,y)|x > 0, y \in \mathbb{R}\}$. For simplicity, we will only consider the case $\mathbb{R}^2$ in the following discussion, but the case $\mathbb{R}^2_{\rightarrow}$ is entirely similar. Since the Lagrangian density does not depend explicitly on the Riemann surface as a consequence of its locality (and the fact that the curvature is zero almost everywhere), it is expected that this partition function can be expressed as an object calculated from a model on $\mathbb{R}^2$, where the structure of the Riemann surface is implemented through appropriate boundary conditions around the points with non-zero curvature. Consider for instance the simple Riemann surface $\mathcal{M}_{n,x_1,x_2}$ with $x_j = (x_j, 0)$, composed of n sheets sequentially joined to each other on the segment $x \in [x_1, x_2]$, $y = 0$

(see figure 2). We would expect that the associated partition function involves certain 'fields'[3] at the points on $\mathbb{R}^2$ given by x_1 and x_2. These fields would be implementing the appropriate non-zero curvature at the branch points.

Expression (2.1) for the partition function essentially defines these fields: it gives their correlation functions, up to a normalization independent of their positions; we only have to put as many branch points as we have such fields in the correlation function. But if we insist in understanding this as the initial model on $\mathbb{R}^2$, this definition makes these special fields non-local. Locality of a field (used here in its most fundamental sense) means that as an observable in the quantum theory, it is quantum mechanically independent of the energy density at space-like distances. In the associated euclidean field theory, this means that correlation functions involving these fields and the energy density are, as functions of the position of the energy density, defined on $\mathbb{R}^2$ and continuous except at the positions of the fields. The energy density is simply obtained from the Lagrangian density. But clearly, bringing the Lagrangian density around a branch point changes the value of the correlation function, since it gets to a different Riemann sheet. Hence, the fields defined by (2.1) seen as fields in a model on $\mathbb{R}^2$ with Lagrangian density $\mathcal{L}[\varphi](x)$ are non-local: the Lagrangian density is not well defined on $\mathbb{R}^2$. Locality is at the basis of most of the results in QFT, so it is important to recover it.

In order to correct the problem, note that if we *defined* a new Lagrangian density on $\mathbb{R}^2$ at the point (x, y) by simply summing the initial Lagrangian density $\sum_{j=1}^{n} \mathcal{L}[\varphi]\big(x_{\mathcal{R}}^{(j)}\big)$ over all the points $x_{\mathcal{R}}^{(j)}$ of the Riemann surface $\mathcal{R} = \mathcal{M}_{n,x_1,x_2}$ that project onto (x, y), then with respect to this new Lagrangian density, the fields defined above would be local. Hence, the idea is simply to consider a larger model: a model formed by n independent copies of the original model, where n is the number of Riemann sheets necessary to describe the Riemann surface by coordinates on $\mathbb{R}^2$. Let us take again the simple example of $\mathcal{M}_{n,x_1,x_2}$. We re-write (2.1) as

$$Z_n(x_1, x_2) \equiv Z\big[\mathcal{L}, \mathcal{M}_{n,x_1,x_2}\big] = \int_{\mathcal{C}(x_1;x_2)} [\mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_n]_{\mathbb{R}^2}\, \mathrm{e}^{-\int_{\mathbb{R}^2} \mathrm{d}^2x (\mathcal{L}[\varphi_1](x)+\cdots+\mathcal{L}[\varphi_n](x))}, \tag{2.2}$$

where $\mathcal{C}(x_1, x_2)$ are *continuity conditions* on the fields $\varphi_1, \ldots, \varphi_n$ restricting the path integral:

$$\mathcal{C}(x_1; x_2) \quad : \quad \varphi_i(x, 0^+) = \varphi_{i+1}(x, 0^-), \qquad x \in [x_1, x_2], \qquad i = 1, \ldots, n, \tag{2.3}$$

where we identify $n + i \equiv i$. What appears in the action of that path integral can now be seen as the Lagrangian density of the multi-copy model,

$$\mathcal{L}^{(n)}[\varphi_1, \ldots, \varphi_n](x, y) = \mathcal{L}[\varphi_1](x, y) + \cdots + \mathcal{L}[\varphi_n](x, y).$$

The energy density of that model is the sum of the energy densities of the n individual copies. Expression (2.2) with (2.3) does indeed define the insertion of local fields at x_1 and x_2 in the multi-copy model, since the energy density is the same on both sides of the segment $x \in [x_1, x_2]$, $y = 0$ according to the conditions $\mathcal{C}(x_1, x_2)$.

The local fields defined by (2.3) are examples of 'twist fields'. Twist fields exist in a QFT model whenever there is a global internal symmetry σ (a symmetry that acts the same way everywhere in space and that does not change the positions of fields): $\int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}y\, \tilde{\mathcal{L}}[\sigma\varphi](x, y) = \int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}y\, \tilde{\mathcal{L}}[\varphi](x, y)$ for some Lagrangian density $\tilde{\mathcal{L}}[\varphi](x)$. Their correlation functions can be formally defined through the path integral:

$$\langle \mathcal{T}_\sigma(x) \cdots \rangle_{\tilde{\mathcal{L}},\mathbb{R}^2} \propto \int_{\mathcal{C}_\sigma(x)} [\mathrm{d}\varphi]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}y\, \tilde{\mathcal{L}}[\varphi](x, y)\right] \ldots, \tag{2.4}$$

[3] Here, the term 'field' is taken in its most general QFT sense: it is an object of which correlation functions—multi-linear maps—can be evaluated, and which depends on a position in space—parameters x, y that transform like coordinates under translation symmetries.

where $\cdots$ represent insertions of other local fields at different positions. The path integral continuity conditions produce a cut on a half-line starting at the point $x = (x, y)$:

$$\mathcal{C}_\sigma(x) \quad : \quad \varphi(x', y^+) = \sigma\varphi(x', y^-), \qquad x' \in [x, \infty). \tag{2.5}$$

The proportionality constant is an infinite constant that is independent of the position x and of those of the other local fields inserted, present in order to render the path integral finite (so that it may represent a correlation function). For insertion of many twist fields, we just add more continuity conditions on different half-lines, starting at different points. The fact that σ is a symmetry ensures that $\mathcal{T}_\sigma$ is local, since it ensures that the energy density is continuous through the cut produced by the continuity condition. Also, it ensures that the result is in fact invariant under continuous changes of the shape of this cut, up to symmetry transformations σ of the local fields that are being swept. This is because inside a loop, we can always apply a symmetry transformation without changing the result of the path integral, up to transformations of local fields present inside the loop. In this way, we can modify the continuity conditions through the loop. By drawing a line, where the usual continuity holds, starting and ending on a twist-field cut (or possibly on its end-points), we form a loop with the twist-field cut. Applying a symmetry transformation inside this loop, we erase part (or all) of the twist-field cut and make the line a new twist-field cut. In this way, it is possible to move the twist-field cut to any shape. Hence our choice of a half-line extending to the right in the definition of the twist field is for convenience.

A consequence of the formal definition (2.4) is that correlation functions $\langle \mathcal{T}_\sigma(x)\mathcal{O}(y)\cdots\rangle_{\tilde{\mathcal{L}},\mathbb{R}^2}$ with some local fields $\mathcal{O}(y)$ are defined, as functions of y (continuous except at positions of other local fields), on a multi-sheeted covering of $\mathbb{R}^2$ with a branch point at $y = x$, whenever $\sigma\mathcal{O} \neq \mathcal{O}$. More precisely, twist fields have the property that a clockwise continuous displacement of a local field $\mathcal{O}(y)$ around x back to its initial projected point on $\mathbb{R}^2$ is equivalent to the replacement $\mathcal{O} \mapsto \sigma\mathcal{O}$ in any correlation function. If $\sigma\mathcal{O} \neq \mathcal{O}$, then $\mathcal{O}$ is said to be 'semi-local' with respect to $\mathcal{T}_\sigma$. This twist-field property is satisfied by a large family of fields, not only the one obtained through the formal definition (2.4). For instance, we could have inserted a field $\varphi(x)$ in the path integral, leading to the same twist property (this is a descendent twist field). However, the twist property, along with the condition that $\mathcal{T}_\sigma$ has the lowest scaling dimension and be invariant under all symmetries of the model that commute with σ (that is, that it be a primary field in the language of conformal field theory), uniquely fixes the field $\mathcal{T}_\sigma$ up to a normalization. These conditions lead to a definition that is in agreement with the path-integral definition (2.4). In fact, these conditions constitute a more fundamental way of defining the primary twist field than the path integral, as they do not require the existence of a Lagrangian density. In particular, they lead to unambiguous definitions in any quantization scheme (we will discuss what the twist condition is in the quantization on the line when we discuss form factors). We will take this general point of view in the following, but we will continue to think of a model of QFT through its Lagrangian density $\tilde{\mathcal{L}}$ for clarity.

Twist fields associated with internal symmetries have been largely studied in the context of CFT: they correspond to twisted modules for vertex operator algebras [56–58], at the basis of so-called orbifold models. In the context of massive integrable QFT, only the simplest (standard) cases are well known. The Ising order and disorder fields are $\mathbb{Z}_2$ twist fields in the free massive Majorana fermion theory, and in the equivalence between the massive Thirring model and the sine-Gordon model, the bosonic exponential fields of the latter are $U(1)$ twist fields of the former. But in fact, for fields with more general twist properties, the form factor equations of integrable QFT were written in [59]. The first extensive study of form factors of non-standard twist fields (see below) was done in [30, 31], and the first time twist fields form

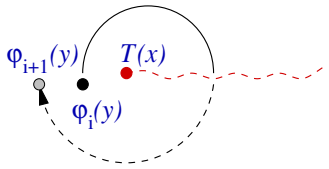

Figure 3. The effect of $\mathcal{T}$ on other local fields.

factors were considered beyond integrability was in [52]; we will describe these works in the following section.

In the n-copy model with Lagrangian $\tilde{\mathcal{L}} = \mathcal{L}^{(n)}$, there is a symmetry under exchange of the copies. The twist fields defined by (2.2), which we call *branch-point twist fields*, are twist fields associated with the two opposite cyclic permutation symmetries $i \mapsto i+1$ and $i+1 \mapsto i$ ($i = 1, \ldots, n, n+1 \equiv 1$). We will denote them simply by $\mathcal{T}$ and $\tilde{\mathcal{T}}$, respectively:

$$\mathcal{T} = \mathcal{T}_\sigma, \qquad \sigma : i \mapsto i+1 \bmod n$$
$$\tilde{\mathcal{T}} = \mathcal{T}_{\sigma^{-1}}, \qquad \sigma^{-1} : i+1 \mapsto i \bmod n$$

(see figure 3 for the case $\mathcal{T}$). More precisely, we have

$$Z_n(x_1, x_2) \propto \langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}. \tag{2.6}$$

This can be seen to be correct by observing that for $x \in [x_1, x_2]$, consecutive copies are connected through $y = 0$ due to the presence of the cut produced by $\mathcal{T}(x_1)$, whereas for $x > x_2$, the additional cut produced by $\tilde{\mathcal{T}}(x_2, 0)$ cancels this, and copies are connected to themselves through $y = 0$. Hence, indeed only a finite cut between x_1 and x_2 remains. The precise proportionality constant was discussed around equation (1.6).

More generally, the identification holds for correlation functions in the model $\mathcal{L}$ on $\mathcal{M}_{n,x_1,x_2}$, this time with an equality sign:

$$\langle \mathcal{O}(y_{\mathcal{R}} \text{ on sheet } i)\cdots\rangle_{\mathcal{L},\mathcal{M}_{n,x_1,x_2}} = \frac{\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\mathcal{O}_i(y)\cdots\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}}{\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}}, \tag{2.7}$$

where $\mathcal{O}_i$ is the field in the model $\mathcal{L}^{(n)}$ coming from the ith copy of $\mathcal{L}$ and y is the projection of $y_{\mathcal{R}}$ onto $\mathbb{R}^2$.

Note that it is easy to transform the twist field $\mathcal{T}$ into $\tilde{\mathcal{T}}$, and vice versa: we only have to apply the 'flip' symmetry transformation by which the order of the copies is inverted (this is another element of the permutation symmetry group). Note also that this construction can also be generalized to Riemann surfaces with more branch points is straightforward, but this will not be needed here.

The conformal dimension of branch-point twist fields is an important characteristic of these fields. It was essentially calculated in [24], although branch-point twist fields were not introduced; only the non-local fields discussed above were considered. Here, we reproduce the derivation of [31], which makes explicit reference to the branch-point twist fields, but otherwise follows closely [24].

Consider the model $\mathcal{L}$ to be a conformal field theory (CFT). Then also $\mathcal{L}^{(n)}$ is a CFT. There are n fields $T_j(w)$ in $\mathcal{L}^{(n)}$ that correspond to the holomorphic stress–energy tensors of the n copies of $\mathcal{L}$, and in particular the sum $T^{(n)}(w) = \sum_{j=1}^n T_j(w)$ is the holomorphic stress–energy tensor of $\mathcal{L}^{(n)}$. The central charge of $\mathcal{L}^{(n)}$ is nc, if c is that of $\mathcal{L}$.

Consider the holomorphic stress–energy tensor $T(w)$ in $\mathcal{L}$. We can evaluate the one-point function $\langle T(w)\rangle_{\mathcal{L},\mathcal{M}_{n,x_1,x_2}}$ by making a conformal transformation from z in $\mathbb{R}^2$ to w in $\mathcal{M}_{n,a_1,a_2}$ (here z and w are complex coordinates, and $w_j = x_j + \mathrm{i}y_j$) given by

$$z = \left(\frac{w - w_1}{w - w_2}\right)^{\frac{1}{n}}.$$

We have

$$\langle T(w)\rangle_{\mathcal{L},\mathcal{M}_{n,x_1,x_2}} = \left(\frac{\partial z}{\partial w}\right)^2 \langle T(z)\rangle_{\mathcal{L},\mathbb{R}^2} + \frac{c}{12}\{z, w\}$$

where the Schwarzian derivative is

$$\{z, w\} = \frac{z'''z' - (3/2)(z'')^2}{(z')^2}.$$

Using $\langle T(z)\rangle_{\mathcal{L},\mathbb{R}^2} = 0$, we obtain

$$\langle T(w)\rangle_{\mathcal{L},\mathcal{M}_{n,x_1,x_2}} = \frac{c(n^2-1)}{24n^2}\frac{(w_1 - w_2)^2}{(w - w_1)^2(w - w_2)^2}.$$

Since, by (2.7), this is equal to $\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)T_j(w)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}/\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}$ for all j, we can evaluate the correlation function involving the stress–energy tensor of $\mathcal{L}^{(n)}$ by multiplying by n:

$$\frac{\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)T^{(n)}(w)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}}{\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}} = \frac{c(n^2-1)}{24n}\frac{(w_1 - w_2)^2}{(w - w_1)^2(w - w_2)^2}.$$

From the usual CFT formula for insertion of a stress–energy tensor

$$\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)T^{(n)}(w)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2} = \left(\frac{1}{w - w_1}\frac{\partial}{\partial w_1} + \frac{h_1}{(w - w_1)^2} + \frac{1}{w - w_2}\frac{\partial}{\partial w_2} + \frac{h_2}{(w - w_2)^2}\right)\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}$$

we identify the scaling dimension of the primary fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ (they have the same scaling dimension) using $\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2} = |w_1 - w_2|^{-2d_n}$:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right). \tag{2.8}$$

This dimension is the lowest possible dimension for fields with the branch-point twist property, as it is the dimension of a primary field in this family.

Hence, the branch-point twist fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are unambiguously defined by specifying their associated symmetry, σ and σ^{-1}, namely that they are invariant under other global symmetry transformations that commute with σ, and by specifying their scaling dimension to be d_n (2.8). This is a definition up to normalization; we will come back to the normalization in the following section. There are of course many other twist fields associated with other elements of the permutation symmetry group, and these may have smaller scaling dimension than d_n, but these are not the branch-point twist fields that are used in order to represent the partition function on multi-sheeted Riemann surfaces.

2.2. Bulk entanglement entropy

Partition functions on Riemann surfaces with branch points can be used in order to evaluate the entanglement entropy. We start by considering the bulk case, where the quantum model is on the line, and the corresponding (euclidean) QFT model is on $\mathbb{R}^2$. We are ultimately interested

in the scaling limit of the ground-state entanglement entropy, (1.2). However, we will provide here general arguments valid not only for the ground state, but also for excited states. Hence, in place of $|\mathrm{gs}\rangle$ in (1.2), we will take some arbitrary, possibly excited state $|\psi\rangle$. In the scaling limit, the ground state $|\mathrm{gs}\rangle$ maps to the QFT vacuum state $|0\rangle$, but excited states map to QFT asymptotic states characterized by the particle content and their momenta (in general, we will keep the notation $|\psi\rangle$ for the corresponding state in QFT). We will later specialize $|\psi\rangle$ to the ground state.

The main idea in order to evaluate the entanglement entropy in the scaling limit of a quantum model is to use the replica trick [22–24]; we will follow more precisely the method of [24]. That is, we use identity (1.4), where we start by considering the trace for positive integer n, evaluate it in QFT (in the scaling limit), then take the appropriate analytic continuation in n in order to evaluate the limit of the derivative. Considering positive integer n in QFT is useful, because there, the trace is directly related to a partition function on a multi-sheeted Riemann surface, studied in the previous subsection. The way this works is as follows.

First, in order to construct ρ_A in the scaling limit, consider the QFT Hilbert space as a space of field configurations $\{\varphi(x), x \in \mathbb{R}\}$ on $\mathbb{R}$. A state $|\psi\rangle$ can be written as a linear combination of field configurations on $\mathbb{R}$ where the coefficients are obtained by path integrals on the lower half of the $\mathbb{R}^2$ plane, $\mathbb{R}^2_\downarrow = \{(x, y) \mid x \in \mathbb{R}, y < 0\}$. The boundary condition on $y = 0$ is determined by the coefficient we are looking at, and the asymptotic condition $y \to 0$ is determined by $|\psi\rangle$. That is,

$$|\psi\rangle = \int [\mathrm{d}\phi]_{\mathbb{R}} |\phi\rangle\langle\phi|\psi\rangle, \quad \langle\phi|\psi\rangle = \frac{1}{\sqrt{Z_1}} \int_{\mathcal{C}} [\mathrm{d}\varphi]_{\mathbb{R}^2_\downarrow} \mathrm{e}^{-S_{\mathbb{R}^2_\downarrow}[\varphi]} \tag{2.9}$$

where $S_{\mathcal{M}}[\varphi] = \int_{\mathcal{M}} \mathrm{d}^2x\, \mathcal{L}[\varphi](x)$ is the action of the model on $\mathcal{M}$, and the condition $\mathcal{C}$ is

$$\mathcal{C} : \left\{ \begin{array}{c} \varphi(x, 0) = \phi(x), x \in \mathbb{R} \\ \varphi(x, y \to -\infty) \sim f_\psi(x, y), x \in \mathbb{R} \end{array} \right\}.$$

Here, $f_\psi(x, y)$ represents the asymptotic condition corresponding to the state $|\psi\rangle$; for instance, $f_\psi(x, y) = 0$ for the vacuum state, otherwise it reproduces wave packets corresponding to asymptotic particles. The number Z_1 is the 'partition function' of the QFT model in the state $|\psi\rangle$:

$$Z_1 = \int_{\mathcal{C}'} [\mathrm{d}\varphi]_{\mathbb{R}^2} \mathrm{e}^{-S_{\mathbb{R}^2}[\varphi]},$$

where

$$\mathcal{C}' : \left\{ \begin{array}{c} \varphi(x, y \to -\infty) \sim f_\psi(x, y), x \in \mathbb{R} \\ \varphi(x, y \to \infty) \sim f^*_\psi(x, y), x \in \mathbb{R} \end{array} \right\}.$$

It is a true partition function if $|\psi\rangle$ is the ground state; otherwise, it corresponds to an excited-state normalization. The factor $1/\sqrt{Z_1}$ is inserted here in order that we maintain the proper normalization $\langle\psi|\psi\rangle = 1$ (the states $|\phi\rangle$ are taken with the natural delta-functional normalization). Note that the coefficients in the expansion of the dual vector $\langle\psi|$ are simply obtained from (2.9) by complex conjugation, along with the inversion of the imaginary time $y \mapsto -y$—this naturally gives a path integral on the other half of $\mathbb{R}^2$.

Next, the density matrix ρ_A is constructed by tracing $|\psi\rangle\langle\psi|$ over degrees of freedom on $\bar{A}$. We sum the diagonal matrix elements of $|\psi\rangle\langle\psi|$, but only in the space $\bar{\mathcal{A}}$, formed by field configurations on $\bar{A}$. This has the effect of 'connecting', on the part $\bar{A}$ of their boundaries, the two surfaces on which the path integrals used to represent matrix elements of $|\psi\rangle$ and $\langle\psi|$ are defined. Hence, we get a path integral on the whole $\mathbb{R}^2$, with continuity on $\bar{A}$, but with an

open slit on A. We are left with a matrix element of ρ_A, determined by field configurations on both sides of the slit on A. That is, we find

$$\langle \phi_A|\rho_A|\phi'_A\rangle = \int [\mathrm{d}\phi_{\bar{A}}]_{\bar{A}} \langle \phi_{\bar{A}}, \phi_A|\psi\rangle\langle\psi|\phi_{\bar{A}}, \phi'_A\rangle = \frac{1}{Z_1}\int_{\mathcal{C}''} [\mathrm{d}\varphi]_{\mathbb{R}^2\setminus(A,0)}\, \mathrm{e}^{-S_{\mathbb{R}^2\setminus(A,0)}[\varphi]} \tag{2.10}$$

where

$$\mathcal{C}'' = \left\{ \begin{array}{c} \varphi(x,0^-) = \phi_A(x), x\in A; \varphi(x,0^+) = \phi'_A(x), x\in A \\ \varphi(x, y\to-\infty)\sim f_\psi(x,y), x\in\mathbb{R}; \varphi(x,y\to\infty)\sim f^*_\psi(x,y), x\in\mathbb{R} \end{array} \right\}.$$

The nth power of ρ_A,

$$\langle \phi_A|\rho_A^n|\phi'_A\rangle = \int [\mathrm{d}\phi_1\, \mathrm{d}\phi_2\cdots\mathrm{d}\phi_{n-1}]_A \langle\phi_A|\rho_A|\phi_1\rangle\langle\phi_1|\rho_A|\phi_2\rangle\cdots\langle\phi_{n-1}|\rho_A|\phi'_A\rangle,$$

can be obtained by taking n copies of ρ_A with their own independent path integrals over φ_j, $j=1,\ldots,n$, and by connecting in a sequential way one side of the slit on A of one copy to the other side of the slit on A of the next copy (that is, having continuity of the fields φ_j through the slit A in this sequential way). Finally, taking the trace connects the last copy to the first, so that we obtain the partition function on a multi-sheeted Riemann surface. Taking A to consist of only one interval for simplicity, with end-points at positions x_1 and x_2 in $\mathbb{R}^2$, we then find, in agreement with (1.5),

$$\mathrm{Tr}_{\mathcal{A}}\rho_A^n = \frac{Z_n(x_1,x_2)}{Z_1^n} \tag{2.11}$$

where the partition function is (2.2). In this derivation, we referred to the definition of the QFT model given by its Lagrangian density $\mathcal{L}$, but as mentioned in the previous subsection, this is just for clarity of exposition; the existence or not of a Lagrangian density does not affect any of the results.

As we saw in the previous subsection, the partition function above can be computed as a two-point correlation function of local fields in the model given by the Lagrangian density $\mathcal{L}^{(n)}$ using (2.6). In the case of the ground state, using (1.6), we have

$$S_A^{\mathrm{bulk}}(r) = -\lim_{n\to1}\frac{\mathrm{d}}{\mathrm{d}n}\mathcal{Z}_n\varepsilon^{2d_n}\langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|0\rangle. \tag{2.12}$$

In the case where $|\psi\rangle$ is an excited state, relation (2.6) still holds, because the derivation of the previous subsection did not make any reference to the asymptotic conditions on the field $\tilde{\varphi}$ as $y\to\pm\infty$. However, in (2.6), we have to understand the correlation function as an excited-state diagonal matrix element of the product of twist fields, in order to implement the appropriate asymptotic conditions in the path integral. Hence, what replaces (1.6) is

$$\frac{Z_n(x_1,x_2)}{Z_1^n} = \mathcal{Z}_n\varepsilon^{2d_n}\langle\psi|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|\psi\rangle, \tag{2.13}$$

so that we obtain

$$S_A^{\mathrm{bulk}}(r) = -\lim_{n\to1}\frac{\mathrm{d}}{\mathrm{d}n}\mathcal{Z}_n\varepsilon^{2d_n}\langle\psi|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|\psi\rangle. \tag{2.14}$$

In this formula, the normalization of $|\psi\rangle$ is not the standard one in the case of an excited state, because of the normalization factor $1/Z_1^n$ in (2.13). Indeed, this factor normalizes away the infinite-volume divergencies coming from colliding rapidities of asymptotic states in the disconnected terms of the two-point function. We will not discuss further here the case of excited states, and concentrate solely on the ground state entanglement entropy.

2.3. From bulk to boundary entanglement entropy

We are now interested in the situation where the system is on the half-line $x > 0$ and the connected region A has an end-point on the boundary of the system $x = 0$. However, it will be most easy and instructive to start with a slightly different problem, where the region A lies entirely in the bulk, with two boundary points on the half line $x > 0$. Since the system is on the half-line, one needs to provide an additional boundary condition at $x = 0$ in order to fully define the model. There are various ways of implementing such a boundary condition. From the viewpoint of the path integral, the boundary condition is implemented by a restriction on the allowed values of the fields $\varphi(x)$ and its derivatives at the boundary $x = 0$, along with, possibly, an extra term S_B in the action that is supported on $x = 0$, $S_B = \int \mathrm{d}y\, \mathcal{L}_B[\varphi](y)$. From the viewpoint of the quantized theory on the half-line $x > 0$, with $y \in \mathbb{R}$ the imaginary time, the boundary condition determines the whole Hilbert space, the vacuum $|0\rangle_B$ and all excited states. Finally, crossing symmetry gives, from the latter, the viewpoint of the quantized theory on the full line $y \in \mathbb{R}$ with imaginary time $x > 0$. There, the boundary condition corresponds to a boundary state $|B\rangle$, a state in the usual Hilbert space on the full line.

The derivations of the previous two subsections, connecting the entanglement entropy to partition functions on multi-sheeted Riemann surfaces, and then connecting the latter to correlation functions of branch-point twist fields, can be directly generalized to the boundary situation. First, retracing the steps of the last subsection in the path-integral formulation of the model on the boundary, we obtain again (2.11), where now the partition functions are path integrals over configurations of the field $\tilde{\varphi}(x)$ with $x \in \mathbb{R}^2_{\rightarrow}$, with a boundary condition at $x = 0$ and possibly an extra boundary term S_B in the action. Second, the derivation of subsection 2.1 can also be directly reproduced, and we obtain, instead of (1.6),

$$\frac{Z_n(x_1, x_2)}{Z_1^n} = \mathcal{Z}_n \varepsilon^{2d_n}\, {}_B\langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|0\rangle_B. \tag{2.15}$$

Here $|0\rangle_B$ is the ground state in the n-copy QFT model on the half line, and $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are the twist fields with the same fundamental definition as before (the twist property, invariance under other symmetry transformations, and lowest scaling dimension), but as operators on the half-line Hilbert space. Hence we obtain

$$S_A^{\text{boundary}}(x_1, x_2) = -\lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n} \mathcal{Z}_n \varepsilon^{2d_n}\, {}_B\langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|0\rangle_B. \tag{2.16}$$

From this, there are two ways to obtain the entanglement entropy $S_A^{\text{boundary}}(r)$ for a region A starting from $x = 0$ and ending at a distance r from it. First, we could consider the limit $x_1 \to (0, 0)$ in $S_A^{\text{boundary}}(x_1, x)$ with $x = (r, 0)$, making the region A approach the boundary. As the twist field at $x_1 = (x_1, 0)$ approaches the boundary, the correlation function diverges, because the presence of the boundary changes the regularization necessary around the branch point. A way to evaluate the divergency is to use boundary conformal field theory, which applies in massive models when a local field is near to a boundary. It tells us that for small x_1 there is a power law determined by the conformal dimension of $\mathcal{T}$: ${}_B\langle 0|\cdots\mathcal{T}(x_1)|0\rangle_B \propto x_1^{-d_n}$ [60]. We may then define $\mathcal{T}(0)|0\rangle_B$ as $\lim_{x_1\to 0} x_1^{d_n}\mathcal{T}(x_1)|0\rangle_B$. This appropriately regularized operator $\mathcal{T}(0)$ is simply proportional to the unitary operator performing a cyclic $\mathbb{Z}_n$ transformation, since its branch cut, through which $\mathbb{Z}_n$ transformations are performed, now extends through the whole space. But since $|0\rangle_B$ is invariant under such a transformation, the action of $\mathcal{T}(0)$ on $|0\rangle_B$ gives $|0\rangle_B$ back. Moreover, since in the definition of $\mathcal{T}(0)$ we already took into account a renormalization of the field, there is no regularization

ε^{d_n} associated with that field in the relation between the partition function and the two-point function. Hence, we find, with appropriate choice of proportionality constants,

$$S_A^{\text{boundary}}(r) = -\lim_{n\to 1}\frac{\mathrm{d}}{\mathrm{d}n}\mathcal{Z}_n\varepsilon^{d_n}\,{}_B\langle 0|\mathcal{T}(x)|0\rangle_B. \tag{2.17}$$

The power of ε guarantees that the scaling dimension of the quantity that is differentiated is 0.

It is important to note that the result of the limit $x_1 \to (0,0)$ is a function only of the distance r between x and the boundary. Indeed, $\mathcal{T}(0)$ is a unitary operator preserving $|0\rangle_B$, hence can be put at any imaginary time without changing the result.

Second, we may take the limit $x_2 \to \infty$ in $S_A^{\text{bulk}}(x, x_2)$, with again $x = (r, 0)$. This should lead to the same result for the entanglement entropy, because of the symmetry $S_A = S_{\bar{A}}$, up to an additive term corresponding to the contribution to the entanglement entropy around the boundary point at ∞. In the limit $x_2 \to \infty$, the two-point function in (2.12) reduces to its disconnected part:

$${}_B\langle 0|\mathcal{T}(x)\tilde{\mathcal{T}}(x_2)|0\rangle_B \sim {}_B\langle 0|\mathcal{T}(x)|0\rangle_B\,{}_B\langle 0|\tilde{\mathcal{T}}(\infty)|0\rangle_B. \tag{2.18}$$

In the second factor, the twist field does not feel the presence of the boundary, hence this expectation value can be replaced by its expectation value in the model without boundary, $\langle 0|\mathcal{T}|0\rangle$. Dividing out this factor (which corresponds to subtracting the contribution to the entanglement entropy around the point at ∞), flipping the sheets in order to transform $\tilde{\mathcal{T}}$ into $\mathcal{T}$, and using the appropriate branch-point regularization, we find again (2.17).

Some of these considerations, and in particular the calculations that we will present below, are made clearer by using crossing symmetry: taking the half-line $x > 0$ to be imaginary time, and the full line $y \in \mathbb{R}$ the space direction. In this picture, as we mentioned, the boundary condition is implemented as a boundary state on the Hilbert space on the full line:

$${}_B\langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|0\rangle_B = \langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|B\rangle. \tag{2.19}$$

The boundary state $|B\rangle$ is in the past at imaginary time $x = 0$ in the Hilbert space of the model on the full line, and the twist fields are placed at imaginary times x_1 and x_2. More precisely, the boundary state is the n-fold tensor product of single-copy boundary states. The state $\langle 0|$ is again the ground state of the n-copy model on the line, corresponding to asymptotic vanishing fields conditions at positive infinite times. No factor occurs in using crossing symmetry since the branch-point twist fields are spinless. The normalization of the boundary state $|B\rangle$ is such that $\langle 0|B\rangle = 1$. By any of the two ways explained above, we obtain for the entanglement entropy:

$$S_A^{\text{boundary}}(r) = -\lim_{n\to 1}\frac{\mathrm{d}}{\mathrm{d}n}\mathcal{Z}_n\varepsilon^{d_n}\langle 0|\mathcal{T}(x)|B\rangle. \tag{2.20}$$

where again the distance between the point x and the boundary is r.

2.4. Normalizations of branch-point twist fields and boundary states

In order to obtain a universal result for the entanglement entropy, we need to fix the short-distance behaviour as in (1.9) in the bulk case, and fix the large-distance behaviour as in (1.11) in the boundary case. As was explained in section 1, this provides a QFT definition for ε (in general, different for the bulk and boundary cases), hence a universal definition for the finite terms in S_A.

The general definition of the branch-point twist fields as explained in subsection 2.1 did not say anything about the normalization of the fields. This normalization does not depend on the positions of the fields, but may, and in general does, depend on the number of copies n—that is, a change of normalization may change the n-dependent number Z_n in (2.12) and

(2.20). Let us choose the *CFT normalization*: this is the normalization by which the short-distance behaviour of the two-point function is exactly given by the CFT two-point function, with coefficient 1:

$$\langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|0\rangle \sim |x_1 - x_2|^{-2d_n} \qquad \text{as} \quad m|x_1 - x_2| \to 0. \tag{2.21}$$

This two-point function just depends on the euclidean distance $|x_1 - x_2|$, thanks to euclidean translation and rotation invariance (or Poincaré invariance in Minkowsky space), and to the fact that the twist fields are spinless. With this normalization, and with a fixed relation between ε and the correlation length ξ, we have a fixed function Z_n in (1.6). In particular, $Z_1 = 1$, because with the CFT normalization, the twist fields just become the identity operator $\mathbf{1}$ at $n = 1$. We may change the definition of ε by rescaling it in an n-independent way, so that we change its relation to the correlation length, but keep it independent from n. This rescaling is absorbed into Z_n, and we may do so in order that $\mathrm{d}Z_n/\mathrm{d}n = 0$ at $n = 1$, as mentioned in section 1. With this rescaling of ε, it is easy to see that taking the derivative in (2.12) with the short-distance behaviour (2.21), we immediately obtain the UV asymptotic given by (1.9) for the entanglement entropy.

Concerning the boundary case, we need to choose the normalization of the boundary state $|B\rangle$ in order that with the CFT normalization (2.21), expression (2.20) gives rise to the correct IR asymptotic in (1.11). In that IR asymptotic, there is the constant U^{model} that appears, a constant that is defined in the bulk situation. At large r, the two-point function in the bulk factorizes into a product of one-point functions (this is asymptotic factorization of correlation functions of local fields):

$$\langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|0\rangle \sim (\langle 0|\mathcal{T}|0\rangle)^2 \qquad \text{as} \quad m|x_1 - x_2| \to \infty, \tag{2.22}$$

where we used the fact that $\tilde{\mathcal{T}}$ is obtained from $\mathcal{T}$ by a permutation symmetry transformation that keeps the vacuum state invariant. Using this into (2.12), we find the correct IR behaviour of (1.9), with

$$U^{\text{model}} = -2 \lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n}(m^{-d_n}\langle 0|\mathcal{T}|0\rangle). \tag{2.23}$$

Hence, this constant is related to the one-point function of twist fields in the CFT normalization. Returning to the boundary case, with the choice of boundary state normalization

$$\langle 0|B\rangle = 1, \tag{2.24}$$

we see that a decomposition into energy and momentum eigenstates, keeping the ground state only, gives

$$\langle 0|\mathcal{T}(x)|B\rangle \sim \langle 0|\mathcal{T}|0\rangle \qquad \text{as} \quad mr \to \infty, \tag{2.25}$$

where r is the distance between x and the boundary. Hence, we indeed find, from (2.20), the correct IR behaviour given by (1.11) for the entanglement entropy in the boundary case.

With this normalization, the model-dependent and boundary-condition-dependent constant $V(\kappa)$ is then universal. Since it occurs at small distances, it is derived from the UV behaviour of the one-point function in (2.20). This behaviour is itself completely controlled by the conformal UV fixed point, hence at short distances that the one-point function is a CFT one-point function. This means that the constant $V(\kappa)$ only depends on the conformal boundary condition to which the system flows as $mr \to 0$. This was seen explicitly in the Ising model in [32], which will be reviewed in section 4.

3. The form factor program for branch-point twist fields

We will now describe how to evaluate correlation functions of branch-point twist fields, involved in the universal part of the entanglement entropy in the bulk and boundary cases through expressions (2.12) and (2.20). The method we will use is that of the form factor expansion; this is a representation of the two-point function (in the bulk case) or the one-point function (in the boundary case) that uses the decomposition of the identity operator into projection operators on energy and momentum eigenstates. It is an effective large-distance (large-r) expansion, and in principle, its re-summation gives a representation valid at all non-zero distances.

3.1. Form factor expansion in (1+1)-dimensional massive QFT

We turn to the description of QFT on the Minkowski spacetime in terms of its Hilbert space of asymptotic relativistic particles. This will allow us to introduce the method we use in order to evaluate the correlation functions of branch-point twist fields: the form factor expansion.

In the context of (1 + 1)-dimensional QFT, form factors are defined as tensor-valued functions representing matrix elements of some local operator $\mathcal{O}(x)$ located at the origin $x = 0$ between a multi-particle *in*-state and the vacuum:

$$F_k^{\mathcal{O}|\mu_1\dots\mu_k}(\theta_1,\dots,\theta_k) := \langle 0|\mathcal{O}(0)|\theta_1,\dots,\theta_k\rangle^{\text{in}}_{\mu_1,\dots,\mu_k}. \tag{3.1}$$

Here $|\theta_1,\dots,\theta_k\rangle^{\text{in}}_{\mu_1,\dots,\mu_k}$ represent the physical 'in' asymptotic states of massive QFT. They carry indices μ_i, which are quantum numbers characterizing the various particle species, and depend on the real parameters θ_i, which are the associated rapidities. The form factors are defined for all rapidities by analytically continuing from some ordering of the rapidities; a fixed ordering provides a complete basis of states, for instance $\theta_1 > \theta_2 > \cdots > \theta_k$. Form factors also depend on the number of copies n, but for simplicity we will not write this dependence explicitly. Moreover, because of relativistic invariance and spinlessness of the branch-point twist fields, they in fact only depend on the rapidity differences; in the two-particle case, they then become functions of only one variable, $\theta = \theta_1 - \theta_2$.

The twist property associated with the permutation symmetry in the general definition of branch-point twist fields was explained in section 2.1. It is at the basis of the main properties of the form factors of branch-point twist fields, and ultimately of our large-distance result about the entanglement entropy. It implies the following fundamental exchange relations for branch-point twist fields as operators on the space of asymptotic states. If $\Psi_1,\dots,\Psi_n$ are the fields associated with the fundamental particles of each copy of the original model, then the equal time ($x^0 = y^0$) exchange relations between $\mathcal{T}$ and $\Psi_1,\dots,\Psi_n$ can be written in the following form[4]:

$$\begin{aligned}
&\Psi_i(y)\mathcal{T}(x) = \mathcal{T}(x)\Psi_{i+1}(y) \qquad && x^1 > y^1,\\
&\Psi_i(y)\mathcal{T}(x) = \mathcal{T}(x)\Psi_i(y) \qquad && x^1 < y^1,
\end{aligned} \tag{3.2}$$

for $i = 1,\dots,n$ and where we identify the indices $n + i \equiv i$. The twist field $\tilde{\mathcal{T}}$ has similar properties as $\mathcal{T}$, with the difference that its exchange relations with the fundamental fields of the theory are given by

$$\begin{aligned}
&\Psi_i(y)\tilde{\mathcal{T}}(x) = \tilde{\mathcal{T}}(x)\Psi_{i-1}(y) \qquad && x^1 > y^1,\\
&\Psi_i(y)\tilde{\mathcal{T}}(x) = \tilde{\mathcal{T}}(x)\Psi_i(y) \qquad && x^1 < y^1,
\end{aligned} \tag{3.3}$$

[4] Here we employ the standard notation in the Minkowski spacetime: x^ν with $\nu = 0, 1$, with x^0 being the time coordinate and x^1 being the position coordinate.

instead of (3.2). The connection with the previous section is obtained by recalling that in going from the Hilbert space description to the path integral description, the order of operators is translated into time-ordering (or y-ordering in euclidean space), and that left-most operators are later in time.

For our present purpose, what is important is that these exchange relations, along with the uniqueness of the branch-point twist fields explained in subsection 2.1, give us the following relation between branch-point twist fields, as operators on the space of asymptotic states:

$$\tilde{\mathcal{T}} = \mathcal{T}^\dagger. \tag{3.4}$$

In order to obtain large-distance expansions, we simply introduce a complete sum over quantum states

$$\mathbf{1} = \sum_{k=1}^{\infty} \sum_{\mu_1 \dots \mu_k} \int_{\theta_1 > \theta_2 > \dots > \theta_k} \frac{\mathrm{d}\theta_1 \dots \mathrm{d}\theta_k}{(2\pi)^k} |\theta_1, \dots, \theta_k\rangle_{\mu_1,\dots,\mu_k \mu_k,\dots,\mu_1} \langle\theta_k, \dots, \theta_1|. \tag{3.5}$$

The two-point function in (2.12) is then expressed as

$$\begin{aligned}\langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|0\rangle = \sum_{k=1}^{\infty} \sum_{\mu_1 \dots \mu_k} \int_{\theta_1 > \theta_2 > \dots > \theta_k} \frac{\mathrm{d}\theta_1 \dots \mathrm{d}\theta_k}{(2\pi)^k} \\ \times \mathrm{e}^{-r \sum_{j=1}^k m_{\mu_j} \cosh\theta_j} \left| \langle 0|\mathcal{T}(0)|\theta_1, \dots, \theta_k\rangle_{\mu_1,\dots,\mu_k} \right|^2\end{aligned} \tag{3.6}$$

where $r = |x_1 - x_2|$ is a spacelike distance between the points x_1 and x_2 in Minkowsky spacetime. Here, we used the fact that $\tilde{\mathcal{T}}$ is the Hermitian conjugate of $\mathcal{T}$ (3.4), and we used translation covariance in space and time in order to extract the exponential factor containing all the position dependence. We also used relativistic invariance and spinlessness to say that the form factors only depend on the rapidity differences. In this way, we could do a simple shift of rapidities in the imaginary direction, so that only $\cosh\theta_j$'s are left in the exponential, with coefficients the negative of the spacelike relativistic distance times the mass of particle μ_j. The integrals, then, become explicitly exponentially convergent (the large-rapidity asymptotics of the form factors is at most exponential in θ_j's). In the same spirit, we could have simply thought of x_1 and x_2 as coordinates in $\mathbb{R}^2$ (with imaginary time), used the euclidean rotation invariance in order to bring $x_1 - x_2$ in the pure imaginary time direction (with a distance r), then used imaginary-time translation covariance to extract the real-exponential factor depending on r. The factor $1/(2\pi)^k$ in the expansion just tells us about the normalization we have chosen for the asymptotic states.

Similarly, the one-point function in (2.20) is expressed as

$$\begin{aligned}\langle 0|\mathcal{T}(x)|B\rangle = \sum_{k=1}^{\infty} \sum_{\mu_1 \dots \mu_k} \int_{\theta_1 > \theta_2 > \dots > \theta_k} \left[\frac{\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_k}{(2\pi)^k} \mathrm{e}^{-r \sum_{j=1}^k m_{\mu_j} \cosh\theta_j} \right. \\ \left. \times \langle 0|\mathcal{T}(0)|\theta_1, \dots, \theta_k\rangle_{\mu_1,\dots,\mu_k \mu_k,\dots,\mu_1} \langle\theta_k, \dots, \theta_1|B\rangle \right].\end{aligned} \tag{3.7}$$

Here, the easiest way to obtain this expansion is by considering that the distance r between x and the boundary is in the imaginary time direction, and using imaginary-time translation covariance in order to bring out the real-exponential factor.

In the expansions (3.6) and (3.7) the terms with a larger number of particles are smaller because the lowest possible value of the argument of the exponential is $-rkm_1$, k times the lowest mass m_1. Moreover, as r increases, the larger particle terms decrease faster; hence, these indeed are large-distance expansions.

3.2. Form factor equations in integrable QFT

We have found that (2.21) and (2.24) provide the correct field and boundary state normalizations in order to reproduce (1.9) and (1.11). We have also explained how the form factors can be used in order to calculate the correlation functions involved in the universal part of the entanglement entropy. Let us now provide more explanation as to the properties of form factors in the context of integrable models of QFT.

The main characteristics of massive integrable models of QFT are that the number of particles and their momenta set are conserved under scattering, and that the scattering matrix factorizes into products of two-particle scattering matrices. The two-particle scattering matrix (or S-matrix) is the solution of a set of consistency equations and analytic properties. These consistency equations and analytic properties are often strong enough, when combined with general properties of the model (symmetries, etc), to completely fix the S-matrix in integrable models. In a similar way, the form factors can be completely fixed by solving a set of equations and analytic properties, which now depend on the two-particle S-matrix.

In this section, we want to show how the standard form factor equations for $(1+1)$-dimensional IQFTs must be modified for the branch-point twist fields. Here we will consider an integrable model consisting of n copies of a known integrable theory possessing a single particle spectrum and no bound states (such as the Ising and sine-Gordon models). The interested reader may refer to [52] for the general case. We have therefore n particles, which we will denote by indices $1, \dots, n$. The S-matrix between particles i and j with rapidities θ_i and θ_j will be denoted by $S_{ij}(\theta_i - \theta_j)$ (that it depends on the rapidity difference is a consequence of relativistic invariance). Particles of different copies do not interact with each other, so that the S-matrix of the model will be of the form

$$S_{ij}(\theta) = S(\theta)^{\delta_{ij}} \qquad \forall\, i, j = 1, \dots, n, \tag{3.8}$$

where $S(\theta)$ is the S-matrix of the single-copy integrable QFT.

It is well known that exchange relations like (3.2) and (3.3) play an important role in the derivation of the consistency equations for the form factors. Generalizing the standard arguments to the exchange relation (3.2), the form factor axioms are

$$F_k^{\mathcal{T}|\dots\mu_i\mu_{i+1}\dots}(\dots,\theta_i,\theta_{i+1},\dots) = S_{\mu_i\mu_{i+1}}(\theta_{ii+1})F_k^{\mathcal{T}|\dots\mu_{i+1}\mu_i\dots}(\dots,\theta_{i+1},\theta_i,\dots), \tag{3.9}$$

$$F_k^{\mathcal{T}|\mu_1\mu_2\dots\mu_k}(\theta_1+2\pi\mathrm{i},\dots,\theta_k) = F_k^{\mathcal{T}|\mu_2\dots\mu_n\hat{\mu}_1}(\theta_2,\dots,\theta_k,\theta_1), \tag{3.10}$$

$$\operatorname*{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mathcal{T}|\bar{\mu}\mu\mu_1\dots\mu_k}(\bar{\theta}_0+\mathrm{i}\pi,\theta_0,\theta_1,\dots,\theta_k) = \mathrm{i}F_k^{\mathcal{T}|\mu_1\dots\mu_k}(\theta_1,\dots,\theta_k), \tag{3.11}$$

$$\operatorname*{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mathcal{T}|\bar{\mu}\hat{\mu}\mu_1\dots\mu_k}(\bar{\theta}_0+\mathrm{i}\pi,\theta_0,\theta_1,\dots,\theta_k) = -\mathrm{i}\prod_{i=1}^{k} S_{\hat{\mu}\mu_i}(\theta_{0i})F_k^{\mathcal{T}|\mu_1\dots\mu_k}(\theta_1,\dots,\theta_k). \tag{3.12}$$

Here $\theta_{ij} = \theta_i - \theta_j$. Besides the simple poles whose residues are given in (3.11) and (3.12), the form factors are, as function of θ_{ij}, analytic in the strip $\mathrm{Im}(\theta_{ij}) \in [0, 2\pi)$ (there would be additional poles, with prescribed residues, if there were bound states; but we only consider integrable models without bound states here). The first axiom is in fact the same as for ordinary local fields. In the second equation, the crossing or locality relation, we introduced the symbols $\hat{\mu}_i = \mu_i + 1$. As compared to the usual form factor equations, it is altered by the nature of the exchange relation and it now relates form factors associated with different particle sets. Finally, the last two equations generalize the standard kinematic residue equation to branch-point twist fields. Once more, the exchange relations (3.2) are responsible for the splitting into two equations. Here, for later convenience, we wrote the equations in their general form

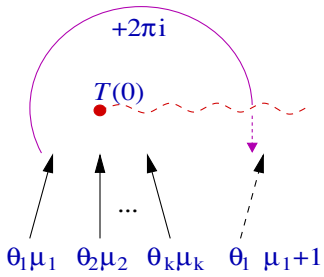

Figure 4. A pictorial representation of the effect of adding $2\pi\mathrm{i}$ to rapidity θ_1 in form factors of the twist field $\mathcal{T}$.

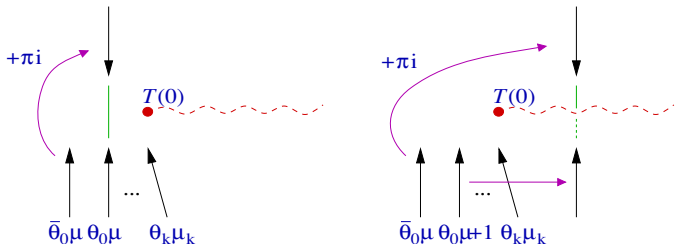

Figure 5. The kinematic poles come from the structure of the wavefunction far from the local fields, at positive and negative infinities. Adding $\mathrm{i}\pi$ to rapidity θ_1 puts the particle in the 'out' region. With a particle in that region, there are delta-functions representing particles in the 'in' region going through without interacting with the local fields. Those occur from the $\mathrm{e}^{\mathrm{i}px}$ form of the wavefunction at positive and negative infinities. But if the coefficients at both limits are different, $S_-\,\mathrm{e}^{\mathrm{i}px}$ and $S_+\,\mathrm{e}^{\mathrm{i}px}$ with $S_- \neq S_+$, then there are also poles in addition to these delta-functions. Only these poles are seen in the analytic continuation $\theta_1 \mapsto \theta_1 + \mathrm{i}\pi$. Different coefficients come from the semi-locality of the twist field and the non-free scattering matrix, as represented here.

valid also for many-particle models, where $\bar{\mu}$ represents the anti-particle associated with μ. In the present case, the integrable model we started with has just one particle (so that μ labels the copies) and therefore each particle is its own anti-particle. Pictorial explanations of the second and of the last two equations are given, respectively, in figures 4 and 5.

3.3. Form factor equations in non-integrable (1+1)-dimensional QFT

Out of integrability, we do not have the luxury of the large simplification of the scattering amplitudes. However, two-particle form factors still satisfy certain properties. For simplicity, we will keep the notation as if there were only one particle type in the original model, so that μ represents the copy number, but attaching an extra index to μ does not affect any of the considerations or equations below.

In the case of an ordinary, spinless local field $\mathcal{O}$, the two-particle form factor $F_2^{\mathcal{O}|\mu_1\mu_2}(\theta_1, \theta_2)$ is a function of $\theta = \theta_1 - \theta_2$ with certain properties when considered as an analytic function of θ, analytically continued from $\theta > 0$ where it is the form factor with 'in' asymptotic states as defined above. The properties of this function are as follows.

- The function analytically continued to negative rapidities $\theta < 0$ along the real line is a form factor with an 'out' asymptotic state:

$$F_2^{\mathcal{O}|\mu_2\mu_1}(\theta_2 - \theta_1) \overset{\theta_1>\theta_2}{=} \langle 0|\mathcal{O}(0)|\theta_1, \theta_2\rangle^{\text{out}}_{\mu_1,\mu_2}. \tag{3.13}$$

- The function analytically continued by a $2\pi\mathrm{i}$ shift from the real line gives

$$F_2^{\mathcal{O}|\mu_1\mu_2}(\theta_1 - \theta_2 + 2\pi\mathrm{i}) = F_2^{\mathcal{O}|\mu_2\mu_1}(\theta_2 - \theta_1). \tag{3.14}$$

- The function is analytic in the physical strip $\mathrm{Im}(\theta) \in [0, 2\pi)$ except possibly for poles on $\mathrm{Re}(\theta) = 0$, $\mathrm{Im}(\theta) \in (0, \pi) \cup (\pi, 2\pi)$ corresponding to bound states.

Note that the physical strip in the rapidity plane, in the last point, is a double covering of the physical sheet in the Mandelstam's s-plane, obtained through $s = m_1^2 + m_2^2 + 2m_1m_2\cosh(\theta_1 - \theta_2)$. This double covering has a symmetry under flip with respect to the point $\mathrm{i}\pi$, as expressed by the second point above. For instance, for $\mathrm{Im}(\theta) = 2\pi$ and $\mathrm{Re}(\theta) > 0$, the function describes a form factor with an 'out' asymptotic state.

The analyticity conditions above are somewhat strong conditions, but they are very far from allowing us to fix the two-particle form factors, because in general, the analytic structure outside of the physical strip is very complicated. However, in the case of the branch-point twist fields, the generalization of these conditions is strong enough to give a result for the entanglement entropy. The generalization is obtained through arguments similar to those presented in the case of integrable models, whereby wave packets are exchanged or rotated around the point where the twist field lies, keeping only in mind that the exchange of wave packets does not lead to a simple two-particle S-matrix in the non-integrable case. The first point above is unchanged by the twist property, as was equation (3.9) in the case of integrable models, because it involves wave packet exchanges only on one copy of the replica model. Since in the replica model, particles on different copies do not interact even in general, non-integrable QFT, we still have here (3.9) in the two-particle case when the copy numbers are different:

$$F_2^{\mathcal{T}|\mu_2\mu_1}(-\theta) = F_2^{\mathcal{T}|\mu_1\mu_2}(\theta) \qquad (\mu_1 \neq \mu_2). \tag{3.15}$$

The second point above is affected by the twist property, in exactly the same way as equation (3.10) was in the integrable case. Hence, equation (3.10) still holds in the non-integrable, two-particle case:

$$F_2^{\mathcal{T}|\mu_1\mu_2}(\theta + 2\pi\mathrm{i}) = F_2^{\mathcal{T}|\mu_2\hat{\mu}_1}(-\theta) \tag{3.16}$$

where we recall that $\hat{\mu} = \mu + 1$. Finally, the third point above holds, up to one additional pole, the kinematic pole, as in the integrable case, equations (3.11) and (3.12). Indeed, the argument for this kinematic pole to be present, given in figure 5, holds true when there is only two particles in the non-integrable case, because there is no exchange of wave packets necessary. Hence, we have

$$\operatorname*{Res}_{\theta=0} F_2^{\mathcal{T}|\bar{\mu}\mu}(\theta + \mathrm{i}\pi) = \mathrm{i}\langle 0|\mathcal{T}|0\rangle, \tag{3.17}$$

$$\operatorname*{Res}_{\theta=0} F_2^{\mathcal{T}|\bar{\mu}\hat{\mu}}(\theta + \mathrm{i}\pi) = -\mathrm{i}\langle 0|\mathcal{T}|0\rangle. \tag{3.18}$$

Equations (3.15)–(3.18) are what is left of (3.9)–(3.12) when integrability is taken away. As we will see below, in the integrable case, and only in this case, we can write exact form factors, but in general, the equations left in the non-integrable case are enough to obtain the exact result displayed in section 1.

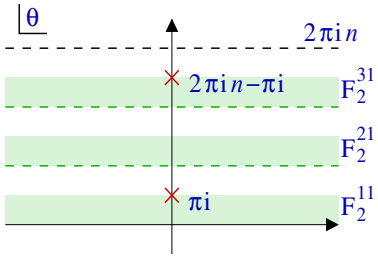

Figure 6. The structure of the function $F_2^{\mathcal{T}|11}(\theta)$ in the extended sheet $\mathrm{Im}(\theta) \in [0, 2\pi n]$, in the case $n = 3$. Crosses indicate the positions of the kinematic singularities. Shaded regions represent the physical sheets of the form factors $F_2^{\mathcal{T}|11}(\theta)$, $F_2^{\mathcal{T}|12}(\theta)$ and $F_2^{\mathcal{T}|13}(\theta)$.

3.4. Two-particle form factors

Let us now analyse the equations for two-particle form factors in more detail. First, it is clear that by a cyclic exchange of the sheets, the branch-point twist fields are unchanged, so that form factors are invariant under such a cyclic exchange. Hence, it is sufficient to look at $F_2^{\mathcal{T}|j1}(\theta)$, for instance. Second, we may repeatedly use (3.16) and (3.15) in order to obtain $F_2^{\mathcal{T}|j1}(\theta)$ for $j = 2, 3, \ldots, n$. For instance,

$$F_2^{\mathcal{T}|11}(\theta + 2\pi\mathrm{i}) = F_2^{\mathcal{T}|12}(-\theta) = F_2^{\mathcal{T}|21}(\theta),$$
$$F_2^{\mathcal{T}|21}(\theta + 2\pi\mathrm{i}) = F_2^{\mathcal{T}|13}(-\theta) = F_2^{\mathcal{T}|31}(\theta),$$

etc. Hence, we find

$$F_2^{\mathcal{T}|ii+k}(\theta) = F_2^{\mathcal{T}|jj+k}(\theta) \qquad \forall\, i, j, k \tag{3.19}$$
$$F_2^{\mathcal{T}|j1}(\theta) = F_2^{\mathcal{T}|11}(\theta + 2\pi i(j-1)) \qquad j = 2, 3, \ldots, n. \tag{3.20}$$

Finally, the residue equations (3.17) and (3.18) as well as the analyticity conditions on the physical strip tell us about the analytic structure of $F_2^{\mathcal{T}|11}(\theta)$ in the larger region $\mathrm{Im}(\theta) \in [0, 2\pi\mathrm{i}n)$, which we call the *extended physical strip* [31]. The function $F_2^{\mathcal{T}|11}(\theta)$ is analytic everywhere in that strip, except for simple poles on $\mathrm{Re}(\theta) = 0$: at $\theta = \mathrm{i}\pi$, with residue $\mathrm{i}\langle 0|\mathcal{T}|0\rangle$, and $\theta = 2\mathrm{i}\pi n - \mathrm{i}\pi$, with residue $-\mathrm{i}\langle 0|\mathcal{T}|0\rangle$, and, possibly, on $0 < \mathrm{Im}(\theta) < \pi$ and $2\pi n - \pi < \mathrm{Im}(\theta) < 2\pi n$ if there are bound states. This structure is depicted in figure 6 for the case $n = 3$ and without bound states.

This structure is all we can say about the two-particle form factors in the non-integrable case. In the integrable case, however, we can say much more; again, we concentrate here on diagonal scattering with a one-particle spectrum and without bound states for simplicity. As usual in the integrable context, we define the minimal form factors $F_{\min}^{\mathcal{T}|jk}(\theta, n)$ to be solutions of equations (3.9) and (3.10) for $k = 2$ without poles in the physical strip $\mathrm{Im}(\theta) \in [0, 2\pi]$. That is,

$$F_{\min}^{\mathcal{T}|kj}(\theta) = F_{\min}^{\mathcal{T}|jk}(-\theta)S_{kj}(\theta) = F_{\min}^{\mathcal{T}|jk+1}(2\pi i - \theta) \qquad \forall\, j, k \tag{3.21}$$

where the S-matrix is given by (3.8). Minimal form factors satisfy the same constraints (3.19) and (3.20), a consequence of which is that the minimal form factor $F_{\min}^{\mathcal{T}|11}(\theta)$ must have no poles in the strip $\mathrm{Im}(\theta) \in [0, 2\pi n]$. From the equations above it is easy to deduce

$$F_{\min}^{\mathcal{T}|11}(\theta) = F_{\min}^{\mathcal{T}|11}(-\theta)S(\theta) = F_{\min}^{\mathcal{T}|11}(-\theta + 2\pi n\mathrm{i}). \tag{3.22}$$

These are the main equations from which we can determine all minimal form factors in integrable models.

In order to develop a systematic procedure to solve these equations, it is useful to recall that, for a standard local operator the minimal form factor equations take the form

$$f_{11}(\theta) = f_{11}(-\theta)S(n\theta) = f_{11}(-\theta + 2\pi \mathrm{i}), \tag{3.23}$$

provided that the S-matrix of the theory is given by $S(n\theta)$. Thus given a solution to the previous equation, the function $F_{\min}^{\mathcal{T}|11}(\theta, n) = f_{11}(\theta/n)$ is automatically a solution to (3.22).

In the context of integrable models, a systematic way of solving such type of equations has been developed whereby, given an integral representation for $S(\theta)$, an integral representation of $f_{11}(\theta)$ can be readily obtained [50]. For diagonal theories, the integral representation of the S-matrix takes the form

$$S(\theta) = \exp\left[\int_0^\infty \frac{\mathrm{d}t}{t} g(t) \sinh\left(\frac{t\theta}{\mathrm{i}\pi}\right)\right], \tag{3.24}$$

where $g(\theta)$ is a function which depends on the theory under consideration. A trivial consequence of the previous equation is

$$S(n\theta) = \exp\left[\int_0^\infty \frac{\mathrm{d}t}{t} g(t/n) \sinh\left(\frac{t\theta}{\mathrm{i}\pi}\right)\right], \tag{3.25}$$

and from here, it is easy to show that

$$f_{11}(\theta) = \mathcal{N} \exp\left[\int_0^\infty \frac{\mathrm{d}t}{t \sinh(nt)} g(t) \sin^2\left(\frac{\mathrm{i}tn}{2}\left(1 + \frac{\mathrm{i}\theta}{\pi}\right)\right)\right], \tag{3.26}$$

where $\mathcal{N}$ is a normalization constant. Therefore, the desired solution is

$$F_{\min}^{\mathcal{T}|11}(\theta) = f_{11}(\theta/n) = \mathcal{N} \exp\left[\int_0^\infty \frac{\mathrm{d}t}{t \sinh(nt)} g(t) \sin^2\left(\frac{\mathrm{i}t}{2}\left(n + \frac{\mathrm{i}\theta}{\pi}\right)\right)\right]. \tag{3.27}$$

In order to evaluate the full two-particle form factors, we must solve (3.21) with the poles in the extended physical strip mentioned before. The 'minimal' solution, with in particular the least diverging behaviour at large θ, is

$$F_2^{\mathcal{T}|jk}(\theta) = \frac{\langle \mathcal{T} \rangle \sin\left(\frac{\pi}{n}\right)}{2n \sinh\left(\frac{\mathrm{i}\pi(2(j-k)-1)+\theta}{2n}\right) \sinh\left(\frac{\mathrm{i}\pi(2(k-j)-1)-\theta}{2n}\right)} \frac{F_{\min}^{\mathcal{T}|jk}(\theta)}{F_{\min}^{\mathcal{T}|jk}(\mathrm{i}\pi)}. \tag{3.28}$$

Note that the constant factor $\sinh(\pi/n)$ guarantees that all form factors vanish for $n = 1$ as expected, since in that case the field $\mathcal{T}$ can be identified with the identity.

For the field $\tilde{\mathcal{T}}$, the exchange relations imply that form factors of the field $\tilde{\mathcal{T}}$ are equal to those of the field $\mathcal{T}$ up to the transformation $i \to n - i$ for each particle i. This means that

$$F_2^{\mathcal{T}|ij}(\theta) = F_2^{\tilde{\mathcal{T}}|(n-i)(n-j)}(\theta). \tag{3.29}$$

This property can be combined with (3.19) and (3.20) to show that

$$F_2^{\tilde{\mathcal{T}}|11}(\theta) = F_2^{\mathcal{T}|11}(\theta), \tag{3.30}$$

$$F_2^{\tilde{\mathcal{T}}|1j}(\theta) = F_2^{\mathcal{T}|11}(\theta + 2\pi \mathrm{i}(j-1)). \tag{3.31}$$

3.4.1. Examples: the Ising and sine-Gordon models. The Ising and sine-Gordon models are examples of the kinds of integrable models for which we have specialized the results of this section: they have a single particle spectrum and no bound states. Their two-particle S-matrices are

$$S_{\rm I}(\theta) = -1 \qquad \text{and} \qquad S_{\rm s-G}(\theta) = \frac{\tanh\frac{1}{2}\left(\theta - {\rm i}\frac{\pi B}{2}\right)}{\tanh\frac{1}{2}\left(\theta + {\rm i}\frac{\pi B}{2}\right)}, \tag{3.32}$$

respectively. In the sine-Gordon model, the parameter $B \in [0, 2]$ is the effective coupling constant which is related to the coupling constant β in the sine-Gordon Lagrangian [61, 62] as

$$B(\beta) = \frac{2\beta^2}{8\pi + \beta^2}, \tag{3.33}$$

under CFT normalization [63]. The S-matrix [64–66] is obviously invariant under the transformation $B \to 2 - B$, a symmetry which is also referred to as a weak-strong coupling duality, as it corresponds to $B(\beta) \to B(8\pi\beta^{-1})$ in (3.33). The point $B = 1$ is known as the self-dual point.

Form factors of local fields of the Ising model were first computed in [50, 67] and later on in [68] for so-called descendant fields. A solution to (3.21) for $j = k = 1$ is given by

$$F_{\rm min}^{\mathcal{T}|11}(\theta) = -{\rm i}\sinh\left(\frac{\theta}{2n}\right), \quad \text{for the Ising field theory.} \tag{3.34}$$

In accordance with our previous arguments, this is the standard minimal form factor already employed in [50, 67], with $\theta \to \theta/n$.

Form factors of the sine-Gordon model were first computed in [69]. The program was thereafter extended to other operators in [70] and more recently in [71]. The S-matrix above admits an integral representation which is given by (3.24), with

$$g(t) = \frac{8\sinh\left(\frac{tB}{4}\right)\sinh\left(\frac{t}{2}\left(1 - \frac{B}{2}\right)\right)\sinh\left(\frac{t}{2}\right)}{\sinh t}. \tag{3.35}$$

Therefore, the minimal form factor is given by

$$F_{\rm min}^{\mathcal{T}|11}(\theta) = \exp\left[-2\int_0^\infty \frac{{\rm d}t\,\sinh\frac{tB}{4}\sinh\frac{t(2-B)}{4}\cosh t\left(n + \frac{i\theta}{\pi}\right)}{t\sinh(nt)\cosh\frac{t}{2}}\right] \times \text{for the sine-Gordon model,} \tag{3.36}$$

where we have chosen the normalization $\mathcal{N} = F_{\rm min}^{\mathcal{T}|11}({\rm i}\pi n)$. This integral representation can be turned into an infinite product of ratios of gamma-functions as described in [31]. As a consistency check, it is quite easy to show that for $n = 1$ the minimal form factor above is the standard minimal form factor associated with local fields in the sine-Gordon model computed in [69]. In appendix A of [31] we showed how the same expression can be derived from the angular quantization scheme [72, 73]. The branch-point twist field minimal form factors were also computed for the sine-Gordon model in [30].

3.5. Higher particle form factors

Form factors with a number of particles higher than 2 can only be dealt with in the integrable case. From the consistency equations for form factors of the twist field $\mathcal{T}$ and generalizing the methods previously used for the computation of form factors of local fields in diagonal theories (see e.g. [69]), it is natural to make the following ansatz for the higher particle form factors:

$$F_k^{\mathcal{T}|11\cdots1}(\theta_1, \ldots, \theta_k) = H_k^{\mathcal{T}|1\cdots1} Q_k^{\mathcal{T}|1\cdots1}(x_1, \ldots, x_k) \prod_{i<j} K(\theta_{ij}), \tag{3.37}$$

where

$$K(\theta_{ij}) = F_2^{\mathcal{T}|11}(\theta_{ij})/\langle\mathcal{T}\rangle, \tag{3.38}$$

and we are considering form factors of k particles of the same type (say type 1), since any other form factors can be obtained from these by repeated use of (3.9) and (3.10). More precisely,

$$F_k^{\mathcal{T}|\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k) = F_k^{\mathcal{T}|11\cdots1}(\theta_1+2\pi\mathrm{i}(\mu_1-1),\theta_2+2\pi\mathrm{i}(\mu_2-1),\ldots,\theta_k+2\pi\mathrm{i}(\mu_k-1)), \tag{3.39}$$

with the ordering $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k$. It is convenient to introduce the variables $x_i = \exp(\theta_i/n)$ with $i = 1,\ldots,k$. Then, exploiting the properties of the two-particle form factor discussed above, we find that the first two form factor consistency equations (3.9) and (3.10) are automatically satisfied, provided that the functions $Q_k^{\mathcal{T}|1\cdots1}(x_1,\ldots,x_k)$ are symmetric in all variables x_i. In other words, $Q_k^{\mathcal{T}|1\cdots1}(x_1,\ldots,x_k)$ can be expressed in terms of elementary symmetric polynomials of the variables x_i. $H_k^{\mathcal{T}|1\cdots1}$ are constants. In particular, $H_0^{\mathcal{T}} = \langle\mathcal{T}\rangle$ and $Q_0^{\mathcal{T}} = 1$. Plugging the ansatz (3.37) on the kinematic residue equations (3.11) and (3.12) produces a set of recursive equations for the constants $H_k^{\mathcal{T}|1\cdots1}$ and the functions $Q_k^{\mathcal{T}|1\cdots1}(x_1,\ldots,x_k)$ which may be solved for specific models.

So far, the only model for which all solutions to these equations are known is the Ising field theory. The solutions were obtained in [32]. Since we are dealing with the free Fermion case, it is natural to expect that the form factors of the twist field would admit closed expressions in terms of Pfaffians, as for the order and disorder fields of the Ising theory. This is indeed the case, and it is easy to show that

$$F_k^{\mathcal{T}|11\cdots1}(\theta_1,\ldots,\theta_k) = \langle\mathcal{T}\rangle\mathrm{Pf}(K), \tag{3.40}$$

where Pf is the Pfaffian, which is mainly characterized by the property that $\mathrm{Pf}(K)^2 = \det(K)$ and K is an anti-symmetric $k \times k$ matrix, with entries

$$K_{ij} = K(\theta_{ij}). \tag{3.41}$$

Note that for the Ising model only k-particle form factors with k even are non-vanishing. The function $K(x)$ has properties

$$K(\theta) = -K(-\theta), \tag{3.42}$$

$$K(\theta)|_{n=1} = 0, \tag{3.43}$$

$$(K(\theta+\mathrm{i}s))^* = -K(\theta-\mathrm{i}s), \qquad \theta, s \in \mathbb{R}, \tag{3.44}$$

where '*' indicates complex conjugation. The Pfaffian expression is nothing else than the application of Wick's theorem on the particles in the asymptotic states (specialized to all particles being on copy 1), a contraction of two particles being $K(\theta_{12})$.

3.6. Form factor consistency checks: the Δ-sum rule

The form factor program above provides form factor solutions, but it is not obvious that these solutions are the correct ones for the branch-point twist fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$. Indeed, as we mentioned, the twist property and invariance under other symmetries alone are not enough to fully fix the field; we need to specify its scaling dimension to be the minimal one. It is generally conjectured that the 'minimal' solution corresponds to a primary field (hence with minimal scaling dimension in a given Virasoro module), but it is always good to verify that the solutions found agree with the expected scaling dimension obtained in the ultraviolet limit.

As is well known, the form factor program provides a way of carrying out this verification by allowing us to compute the correlation functions of various fields of an integrable QFT. In the ultraviolet limit, it is possible to relate a particular correlation function to the holomorphic conformal dimension Δ (with $\Delta + \bar{\Delta} = d$, the scaling dimension, and $\Delta - \bar{\Delta} = s$, the spin) of a primary field by means of the so-called Δ-sum rule:

$$\Delta^{\mathcal{T}} = \Delta^{\tilde{\mathcal{T}}} = -\frac{1}{2\langle\mathcal{T}\rangle}\int_0^\infty r\,\langle\Theta(r)\tilde{\mathcal{T}}(0)\rangle\,\mathrm{d}r \tag{3.45}$$

(where the integration is on a space-like ray), originally proposed by Delfino *et al* [74]. Here, Θ is the local operator corresponding to the trace of the stress–energy tensor. The first equality, naturally expected from CFT, holds from the Δ-sum rule thanks to the fact that Θ commute with $\mathcal{T}$ and that $\Theta^\dagger = \Theta$. In the cases of the branch-point twist fields, which are spinless, the holomorphic conformal dimension is related to the scaling dimension by $d_n = 2\Delta^{\mathcal{T}}$, where d_n is expected to be (2.8).

Introducing the complete sum over quantum states (3.5) and carrying out the r-integration, the expression above can be rewritten as

$$\Delta^{\mathcal{T}} = -\frac{1}{2\,\langle\mathcal{T}\rangle}\sum_{k=1}^{\infty}\sum_{\mu_1\ldots\mu_k}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1\cdots\mathrm{d}\theta_k}{k!(2\pi)^k\big(\sum_{i=1}^k m_{\mu_i}\cosh\theta_i\big)^2} \times F_k^{\Theta|\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k)\big(F_k^{\mathcal{T}|\mu_1\cdots\mu_k}(\theta_1,\ldots,\theta_k)\big)^*, \tag{3.46}$$

where the sum in μ_i with $i = 1,\ldots,k$ is a sum over particle types in the theory under consideration. The sum starts at $k = 1$ since we are considering 'connected' correlation functions, that is, the $k = 0$ contribution has been subtracted. The sum above, can only be carried out in particularly simple cases. For most models, one must be content with evaluating just the first few contributions to the sum. Fortunately, the many studies carried out in the last years provide strong evidence that the sum above is convergent and that in fact, the first few terms provide the main contribution to the final result. Indeed, the convergence is often so good that considering only the contribution with $k = 2$ already provides very precise results (see e.g. [69]). Expecting a similar behaviour also in our case, we will approximate the sum above by the two-particle contribution, that is

$$\Delta^{\mathcal{T}} \approx -\frac{n}{2\,\langle\mathcal{T}\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1\,\mathrm{d}\theta_2 F_2^{\Theta|11}(\theta_{12})F_2^{\mathcal{T}|11}(\theta_{12})^*}{2(2\pi)^2m^2\,(\cosh\theta_1+\cosh\theta_2)^2}. \tag{3.47}$$

The factor of n is a consequence of summing over all particle types and using (3.19). In addition, the only non-vanishing contribution comes from form factors involving only one particle type, since we are considering n non-interacting copies of the model. This implies that

$$F_2^{\Theta|ij}(\theta) = 0 \qquad \forall\, i \neq j. \tag{3.48}$$

Changing variables to $\theta = \theta_1 - \theta_2$ and $\theta' = \theta_1 + \theta_2$ we obtain

$$\Delta^{\mathcal{T}} \approx -\frac{n}{32\pi^2m^2\,\langle\mathcal{T}\rangle}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{F_2^{\Theta|11}(\theta)F_2^{\mathcal{T}|11}(\theta)^*}{\cosh^2(\theta/2)}. \tag{3.49}$$

In [31] we evaluated the integral above both for the Ising and sine-Gordon models, whereas in [30] we performed a similar computation for the sine-Gordon model. We briefly summarize below our results for the Ising and sine-Gordon theories.

3.6.1. The Δ-sum rule for the Ising and sine-Gordon models. For the Ising model the only non-vanishing form factor of the trace of the stress–energy tensor is the two-particle form factor. Hence, the two-particle approximation (3.47) becomes exact. The two-particle form factors are given by

$$F_2^{\mathcal{T}|11}(\theta) = \frac{-\mathrm{i}\langle\mathcal{T}\rangle\cos\left(\frac{\pi}{2n}\right)}{n\sinh\left(\frac{\mathrm{i}\pi+\theta}{2n}\right)\sinh\left(\frac{\mathrm{i}\pi-\theta}{2n}\right)}\sinh\left(\frac{\theta}{2n}\right), \qquad F_2^{\Theta|11}(\theta) = -2\pi\mathrm{i}m^2\sinh\left(\frac{\theta}{2}\right), \tag{3.50}$$

and therefore

$$\Delta^{\mathcal{T}} = -\frac{1}{16\pi}\int_{-\infty}^{\infty}\frac{\cos\left(\frac{\pi}{2n}\right)\sinh\left(\frac{\theta}{2n}\right)\sinh\left(\frac{\theta}{2}\right)}{\sinh\left(\frac{\mathrm{i}\pi+\theta}{2n}\right)\sinh\left(\frac{\mathrm{i}\pi-\theta}{2n}\right)\cosh^2\left(\frac{\theta}{2}\right)}\mathrm{d}\theta. \tag{3.51}$$

The above integral can be computed exactly for n even by shifting t by $2\pi n\mathrm{i}$ and noticing that the integral changes by a sign, so that

$$2\Delta^{\mathcal{T}} = 2\pi\mathrm{i}\sum_{i=1}^{n} r_j = \frac{1}{24}\left(n-\frac{1}{n}\right), \tag{3.52}$$

where r_j are the residues of the poles of the integrand at $t = \mathrm{i}\pi(2j-1)$, with $j = 1,\ldots,n$. These residues can be easily computed (for details see [31]) and the expected result (2.8) is reproduced, with $c = 1/2$. The result (3.52) in fact holds for any n.

For the sine-Gordon model, the relevant two-particle form factors are given by

$$F_2^{\mathcal{T}|11}(\theta) = \frac{\langle\mathcal{T}\rangle\sinh\left(\frac{\pi}{n}\right)}{2n\sinh\left(\frac{\mathrm{i}\pi+\theta}{2n}\right)\sinh\left(\frac{\mathrm{i}\pi-\theta}{2n}\right)}\frac{F_{\min}^{\mathcal{T}|11}(\theta)}{F_{\min}^{\mathcal{T}|11}(\mathrm{i}\pi)}, \qquad F_2^{\Theta|11}(\theta) = 2\pi m^2\left.\frac{F_{\min}^{\mathcal{T}|11}(\theta)}{F_{\min}^{\mathcal{T}|11}(\mathrm{i}\pi)}\right|_{n=1}. \tag{3.53}$$

The form factors of Θ were computed in [69]. Since Θ is a local operator, its minimal form factor is given by (3.36) with $n = 1$. The tables below show the result of carrying out the integral (3.49) numerically for various values of n and B. Next to each value of n in brackets we show for reference the expected value of $\Delta^{\mathcal{T}}$, as predicted by the CFT formula (2.8) (with, again, $\Delta^{\mathcal{T}} = d_n/2$).

	$n = 2$ (0.0625)	$n = 3$ (0.1111)	$n = 4$ (0.1563)	$n = 5$ (0.2)
$B = 0.02$	0.0620	0.1114	0.1567	0.2007
$B = 0.2$	0.0636	0.1135	0.1599	0.2048
$B = 0.4$	0.0636	0.1148	0.1620	0.2074
$B = 0.6$	0.0643	0.1155	0.1631	0.2088
$B = 0.8$	0.0644	0.1158	0.1636	0.2096
$B = 1$	0.0644	0.1159	0.1637	0.2098

	$n = 6$ (0.2431)	$n = 7$ (0.2857)	$n = 8$ (0.3281)	$n = 9$ (0.3704)	$n = 10$ (0.4125)
$B = 0.02$	0.2436	0.2864	0.3289	0.3712	0.4135
$B = 0.2$	0.2488	0.2925	0.3360	0.3793	0.4225
$B = 0.4$	0.2522	0.2966	0.3407	0.3846	0.4284
$B = 0.6$	0.2540	0.2988	0.3433	0.3876	0.4317
$B = 0.8$	0.2550	0.2999	0.3446	0.3890	0.4334
$B = 1$	0.2552	0.3002	0.3449	0.3895	0.4339

The figures obtained are extremely close to their expected value for all choices of B and n.

4. Bulk and boundary entropy from twist field form factors

Formulae (2.12) and (2.20) express the entanglement entropy in the bulk and boundary cases, respectively, in terms of the two-point and one-point functions of branch-point twist fields. The form factor expansions (3.6) and (3.7) then express these correlation functions as infinite series obtained from form factors (3.1). These infinite series provide exact expressions, but also, as we explained in section 3.1, the truncated, partial series give efficient large-distance expansions. Our main result, (1.12), is an exact large-distance correction for the bulk entanglement entropy, and is obtained by keeping, in the series (3.6), only the two-particle contributions. On the other hand, our analysis of the boundary entanglement entropy, in the Ising model, required the consideration of all terms in (3.7). We will now explain how to go from form-factor expansions to entanglement entropy.

4.1. Bulk entanglement entropy

From expressions (3.6) and (2.12), we find for the bulk entanglement entropy:

$$S_A^{\text{bulk}}(rm) = -\frac{c}{3}\log(\varepsilon m) + U^{\text{model}} + \sum_{k=1}^{\infty} e_k(rm), \tag{4.1}$$

where U^{model} was defined in (2.23), and

$$e_k(rm) = -\lim_{n\to 1}\frac{\mathrm{d}}{\mathrm{d}n}\left[\sum_{k=0}^{\infty}\sum_{\mu_1,\dots,\mu_k=1}^{n}\int_{\theta_1>\cdots>\theta_k} \times\left[\prod_{j=1}^{k}\frac{\mathrm{d}\theta_j}{2\pi}\,\mathrm{e}^{-rm_{\mu_j}\cosh\theta_j}\right]\left|F_k^{\mathcal{T}|\mu_1\mu_2\dots\mu_k}(\theta_1,\dots,\theta_k)\right|^2\right]. \tag{4.2}$$

The expression for e_k could be re-written in various ways, by performing the sum in one of particle indices μ_i using invariance under shift of copy numbers, and/or by using formula (3.39) in integrable models. Also, again in integrable models, the range of rapidity integration can easily be extended to $-\infty < \theta_j < \infty$, $j = 1, 2, \dots, k$ (i.e. without ordering of rapidities), putting a factor $1/k!$. Indeed, the exchange of two rapidities in the form factors just corresponds to a multiplication by a two-particle S-matrix, according to (3.9). By unitarity, $|S_{\mu_1\mu_2}(\theta)|^2 = 1$, so that the integrands in (3.6) are the same under exchange of rapidities. Another way of understanding this is that with different ordering of rapidities, we simply have a different basis of states: instead of in-states, we have states where in- and out-configurations are mixed. We may also sum over these states in order to get the form-factor expansion, or we may average over many such bases, integrating in the full rapidity range. Such mixed bases exist as asymptotic-state bases in integrable models thanks to the independence of the scattering amplitudes upon impact parameters (see e.g. [43]). In non-integrable models, mixed bases are not expected to exist. However, thanks to (3.13), the latter argument can still be used, for the two-particle contribution, to extend the integration region to the whole rapidity range, without ordering. This will be useful in the following paragraph, where we study in more details the two-particle contribution.

4.1.1. Next-to-leading order IR correction to the bulk entanglement entropy. From the expression above (4.1), it appears that the first correction to the IR behaviour of the entanglement entropy would come from the one-particle form factor contribution

$$e_1(rm) = -\lim_{n\to 1}\frac{\mathrm{d}}{\mathrm{d}n}\left[n\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{e}^{-rm\cosh\theta}\left|F_1^{\mathcal{T}|1}(\theta)\right|^2\right]. \tag{4.3}$$

For all theories studied in [30, 31] this correction vanishes because their internal symmetries are such that only form factors of even particle numbers are non-vanishing. However, it is possible to argue that even when the one-particle form factor is not zero, its contribution to the entropy should be. First, the one-particle form factor is always independent of the rapidity due to relativistic invariance and the spinlessness of the twist field. Hence, it only gives a simple n-dependent factor. Second, the form factor is zero at $n = 1$, as in that case the twist field is simply the identity. If we assume that it goes to zero like $n-1$ (or in fact, faster than $\sqrt{n-1}$), then we can conclude that the derivative of its square at $n = 1$ must also vanish. Certainly, however, it would be desirable to evaluate exactly one-particle form factors in models where they do not vanish by symmetry argument, in order to verify this.

We will now look at the next correction (the first non-trivial one), namely the correction associated with the two-particle form factor contribution

$$\begin{aligned} e_2(rm) &= -\lim_{n\to 1}\frac{\mathrm{d}}{\mathrm{d}n}\left[n\sum_{j=0}^{n-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{2!(2\pi)^2}\left|F_2^{\mathcal{T}|11}(\theta_{12}+2\pi\mathrm{i}j)\right|^2\mathrm{e}^{-rm(\cosh\theta_1+\cosh\theta_2)}\right] \\ &= -\lim_{n\to 1}\frac{\mathrm{d}}{\mathrm{d}n}\left[\langle\mathcal{T}\rangle^2\frac{n}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}\theta\; f(\theta,n)K_0(2rm\cosh(\theta/2))\right]. \end{aligned} \tag{4.4}$$

In the first line, we extended the rapidity integration range so that there is no ordering of rapidities (dividing by 2!) as described above, we summed over one-particle index using invariance under shift of copy numbers, and we used formula (3.20). Note that these operations are valid both at and out of integrability. In the second line we carried out one of the integrals, where $K_0(x)$ is a modified Bessel function of the second kind, and we define

$$f(\theta,n) = \langle\mathcal{T}\rangle^{-2}\left(\left|F_2^{\mathcal{T}|11}(\theta)\right|^2 + \sum_{j=1}^{n-1}\left|F_2^{\mathcal{T}|11}(\theta+2\pi\mathrm{i}j)\right|^2\right). \tag{4.5}$$

For simplicity, we assumed that there is only one particle type, so that both particles involved in this two-particle contribution have the same mass. As we will see, this turns out to be enough in order to obtain the general result.

What follows will be devoted to show that, if we denote by $\tilde{f}(\theta,n)$ the analytic continuation of $f(\theta,n)$ to non-integer values of n, then

$$-\lim_{n\to 1}\frac{\mathrm{d}}{\mathrm{d}n}[n\tilde{f}(\theta,n)] = -\frac{\pi^2\delta(\theta)}{2}, \tag{4.6}$$

which implies

$$e_2(rm) = -\frac{K_0(2mr)}{8}. \tag{4.7}$$

Recall that $f(\theta,1) = 0$ and that $f(\theta,n)$ is only defined for integer values of n. It is also easy to see, from the general solution for the two-particle form factors (3.28) or from the general expectation that the form factors vanish like $n-1$ as $n\to 1$, that the term $\left|F_2^{\mathcal{T}|11}(\theta)\right|^2$ will not contribute to the derivative at $n = 1$.

In order to obtain the entropy, we should now analytically continue the expression inside the derivative in (4.4), as function of rm and n, from $n = 1, 2, 3, \ldots$ to $n \in [1,\infty)$, compute

the derivative with respect to n and evaluate the result at $n = 1$. The analytic continuation is of course not unique. The motivation for the choice of analytic continuation that was used in [30, 31] was based on Carlson's theorem, and on an expectation about the large-n behaviour of partition functions on multi-sheeted Riemann surfaces (or spaces with conical singularities). Essentially, we want the analytic continuation of $f(\theta, n)$, for any fixed θ, to be well-behaved enough at large n. In the integrable cases studied with precision in [30, 31], we found that there was an analytic continuation such that the divergence at large n is less than exponential for any $\mathrm{Re}(n) > 0$. This is a unique analytic continuation with this property by Carlson's theorem.

Once we admit this requirement on the analytic continuation, the evaluation of the derivative at $n = 1$ can be done exactly. The observation at the basis of this evaluation is that although the analytic continuation of the sum in (4.5) vanishes as $n \to 1$ for all $\theta \neq 0$, it does not vanish as $n \to 1$ when $\theta = 0$. That is, the function $\tilde{f}(\theta, n)$ does not convergence uniformly in θ as $n \to 1$. Let us define for convenience

$$\tilde{f}(n) := \tilde{f}(0, n). \tag{4.8}$$

What we find is that its limit $n \to 1$ is non-zero and positive. Note that this means that in fact, $f(0, n)$ does not have a well-behaved analytic continuation from $n = 1, 2, 3, \ldots$ to $n \in [1, \infty)$. Rather, $\tilde{f}(n)$ is such an analytic continuation, but from $n = 2, 3, \ldots$ to $n \in [1, \infty)$; it has the property that $\tilde{f}(1) \neq f(0, 1) = 0$. This was observed explicitly in the Ising and sine-Gordon model [31], where we found $\tilde{f}(n) = 1/2$ in the Ising model, and obtained a large-n expansion for $\tilde{f}(n)$ in the sine-Gordon model from which we could evaluate with some precision its value at $n = 1$. It was also observed in the sine-Gordon model [30], using this time a power series in n. See figure 7.

Naturally, the fact that the function $\tilde{f}(\theta, n)$ does not converge to 0 as $n \to 1$ at the point $\theta = 0$ is of no importance if we just want to evaluate the integral over θ (with, as factor, a function that is well behaved at $\theta = 0$). Indeed, the point $\theta = 0$ is of Lebesgue measure zero, so the value of the non-uniform limit at that point, if it is finite, does not affect the integral. However, we want to take the derivative with respect to n. It is possible to argue that this derivative $\frac{\partial}{\partial n}\tilde{f}(\theta, n)$ at $n = 1$ should actually be proportional to a *delta-function* $\delta(\theta)$: it has non-zero support at $\theta = 0$. Intuitively, this is because looking at the value of $\tilde{f}(\theta, n)$ near enough to $\theta = 0$, we see that it should vary very fast as $n \to 1$, since it should go to zero for any $\theta \neq 0$, it should stay finite for $\theta = 0$, and it should be continuous in θ for all $n > 1$. This very fast variation becomes an infinite variation as $n \to 1$ for θ infinitesimally near to 0, which produces a $\delta(\theta)$. Note that the resulting derivative at $n = 1$ should also be positive, since this fast variation goes from $\tilde{f}(0) > 0$ to 0 as n decreases to 1. This gives a negative correction to the saturation of the entanglement entropy, which is indeed expected on physical grounds.

But why would the function $\tilde{f}(\theta, n)$, as function of θ, not converge uniformly as $n \to 1$? In order to answer this, we have to find a way of analytically continuing in n the summation term in (4.5). There are a great many ways of doing this, and in all cases, what we find is that the cause for the non-uniform convergence is the collision of kinematic poles, at $\theta = \mathrm{i}\pi$ and $\theta = 2\mathrm{i}\pi n - \mathrm{i}\pi$, as $n \to 1$, see figure 8. For instance, using a Poisson resummation formula to transform the sum over j into an integral [31], one sees that the kinematic poles pinch this integral as $n \to 1$. Another way is to perform the summation via a contour integral with a cotangent kernel whose residues reproduce the terms that are to be summed. There, one has to extract the poles of the form factors themselves, these extracted poles collide as $n \to 1$. Both ways necessitate periodicity of the function $\tilde{f}(\theta, n)$ upon adding $2\mathrm{i}\pi n$ to θ. The latter way was in fact generalized to non-integrable models in [52].

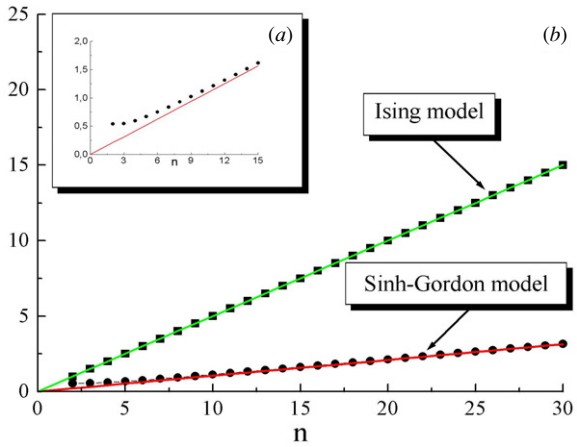

Figure 7. Part (a) shows four functions: the points are the function $nf(0, n)$ for integer values of n in the interval [2, 30] both for the Ising (squares) and sine-Gordon (circles) models, evaluated numerically. The solid line crossing the squares gives the analytic continuation $n\tilde{f}(n)$ for real values of n in the interval [0, 30] for the Ising model, that is the function $n/2$. Note that $\tilde{f}(1) = 1/2$ whereas $f(0, 1) = 0$. Finally, the solid line crossing the circles gives the function $n\tilde{f}(\infty)$ for the sine-Gordon model, that is a straight line passing through the origin which describes the asymptotic behaviour of the function $n\tilde{f}(n)$ for n large. In the sine-Gordon case, all functions have been computed for $B = 0.5$. Part (b) is a magnification of the lower left corner of the sine-Gordon part of (a) which demonstrates (4.11), namely, as for the Ising model, $\tilde{f}(1) = 1/2$ and $f(0, 1) = 0$. For the sine-Gordon model, the analytic continuation is $n\tilde{f}(n) = a_0 n + \sum_{i=1}^{\infty} \frac{a_i}{n^{2i-1}}$, for some coefficients $a_0, a_1, \ldots$ which depend on the coupling B [31]. Such analytic continuation was also studied for the sine-Gordon model in [30] and found to display a power series structure, with powers depending on the values of the coupling constant, very different from the one identified for the sine-Gordon model.

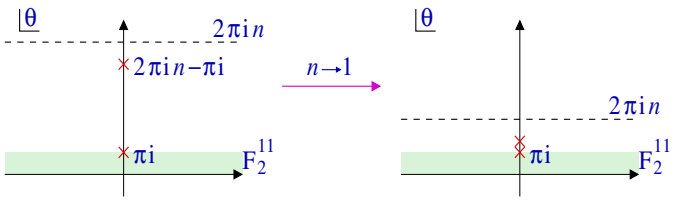

Figure 8. The collision of kinematic singularities when $n \to 1$.

Yet, perhaps the simplest way of extracting the derivative is simply to extract the kinematic poles themselves and then to perform the summation exactly on these poles [31]. The contribution of the kinematic singularities to the sum in the function $\tilde{f}(\theta, n)$ (4.5) is obtained from the singular behaviour in j of the summand $s(\theta, j) = F_2^{\mathcal{T}|11}(\theta+2\pi\mathrm{i}j)\big(F_2^{\mathcal{T}|11}\big)^*(\theta-2\pi\mathrm{i}j)$:

$$s(\theta, j) \sim \frac{\mathrm{i}F_2^{\mathcal{T}|11}(2\theta + 2\pi\mathrm{i}n - \mathrm{i}\pi)}{\theta - 2\pi\mathrm{i}j + 2\pi\mathrm{i}n - \mathrm{i}\pi} - \frac{\mathrm{i}F_2^{\mathcal{T}|11}(2\theta + \mathrm{i}\pi)}{\theta - 2\pi\mathrm{i}j + \mathrm{i}\pi} + \text{c.c.}$$

(where c.c. means complex conjugate, taken for real θ and real j). It is a simple matter to perform on this expression the sum $\sum_{j=1}^{n-1}$, giving

$$\sum_{j=1}^{n-1} s(\theta, j) \sim \frac{1}{2\pi}\left(\psi\left(-\frac{1}{2}+n-\frac{\mathrm{i}\theta}{2\pi}\right)-\psi\left(\frac{1}{2}-\frac{\mathrm{i}\theta}{2\pi}\right)\right) F_2^{\mathcal{T}|11}(2\theta+2\pi\mathrm{i}n-\mathrm{i}\pi)$$
$$+\frac{1}{2\pi}\left(\psi\left(-\frac{1}{2}+n+\frac{\mathrm{i}\theta}{2\pi}\right)-\psi\left(\frac{1}{2}+\frac{\mathrm{i}\theta}{2\pi}\right)\right) F_2^{\mathcal{T}|11}(2\theta+\mathrm{i}\pi)$$
$$+\text{c.c.},$$

where $\psi(z) = \mathrm{d}\log\Gamma(z)/\mathrm{d}z$ is the derivative of the logarithm of Euler's gamma function. This has no poles at $\theta = 0$, as the kinematic poles of the form factors involved that would produce a pole at $\theta = 0$ actually cancel out. The poles that are nearest to $\mathrm{Re}(\theta) = 0$ as $n \to 1$ are at $\theta = \pm\mathrm{i}\pi(n-1)$, again coming from the form factors involved. The residues to first order in $n-1$ give

$$\tilde{f}(\theta, n) \sim \tilde{f}(1)\left(\frac{\mathrm{i}\pi(n-1)}{2(\theta+\mathrm{i}\pi(n-1))}-\frac{\mathrm{i}\pi(n-1)}{2(\theta-\mathrm{i}\pi(n-1))}\right) \quad (n \to 1) \tag{4.9}$$

with

$$\tilde{f}(1) = \tfrac{1}{2}. \tag{4.10}$$

This has simple poles at $\theta = \pm\mathrm{i}\pi(n-1)$ with residues that vanish at $n = 1$, gives $\lim_{n\to 1}\tilde{f}(0, n) = \tilde{f}(1)$, and vanishes like $(n-1)^2$ as $n \to 1$ for $\theta \neq 0$. The limit $n \to 1$ of the derivative with respect to n, as a distribution on θ, is easily evaluated:

$$\left(\frac{\partial}{\partial n}\tilde{f}(\theta, n)\right)_{n=1} = \pi^2\tilde{f}(1)\delta(\theta). \tag{4.11}$$

The full form of $\tilde{f}(\theta, n)$ was obtained in appendix C of [31]. For the free case our result is in agreement with the $n \to 1$ limit evaluated previously in [75].

Inserting this inside (4.1) we find that the entanglement entropy $S_A(rm)$ for A an interval of length r is

$$S_A^{\mathrm{bulk}}(rm) = -\frac{c}{3}\log(\varepsilon m) + U^{\mathrm{model}} - \frac{1}{8}K_0(2rm) + O(\mathrm{e}^{-3rm}), \tag{4.12}$$

with U^{model} defined in (2.23). This result can be generalized to theories with more than one particle and/or bound states [30]: the bound state poles never collide, hence do not provide additional contributions; and the kinematic poles only occur for particle–antiparticle form factors, so that we have a contribution for all particle types of the theory. The result was also generalized to non-integrable models: the only properties needed to derive the result were those seen in section 3.3 to hold out of integrability. Hence, we obtain the more general result (1.12) quoted in section 1.

In general, the constant U^{model} is extremely hard to evaluate, even in integrable models. However, for models of free particles like the Ising model, it can be evaluated by standard methods of angular quantization [76, 77]. This was done in [31] for the Ising model (by simply relating the branch-point twist fields to ordinary $U(1)$-twist fields for which the calculation had already been done), where we obtained the value (1.10).

A final note is in order concerning the function $\tilde{f}(n)$. Above we concentrated on its behaviour as $n \to 1$, as this is what matters for the evaluation of the entanglement entropy. However, the behaviour of this function as $n \to \infty$ is also somewhat interesting. Indeed, in all models studied where the perturbation from the UV fixed point was strictly relevant (Ising, sine-Gordon and sine-Gordon models), the behaviour at large n was observed to be exactly linear. However, for the sine-Gordon model at a value of the coupling that makes the perturbation marginally relevant, we observed a behaviour like $n\log n$ [30]. Interestingly, then, it seems that the large-n behaviour encodes information about the short-distance theory: the way the model behaves near to the critical point.

4.1.2. Higher order IR corrections to the bulk entanglement entropy. In order to evaluate higher order corrections to the entropy, namely $e_3(mr)$, $e_4(mr)$, ... we need first of all to know the form factors for more than two particles. As indicated in previous sections, such solutions are only known for the Ising model (see (3.40)) for which all higher particle corrections to the entropy were evaluated in [32]. Once more, the problem of finding the right analytic continuation in n plays a fundamental role. Solving this problem has been indeed, even for the Ising model, extremely challenging (see [32] for the details of this). Although we have not investigated interacting models yet, we suspect that higher order corrections may also be, at least to some extent, of a universal nature, as the pole structure of the form factors plays again a key role. For the Ising model we have found that

$$e_{2k}(rm) = \frac{\pi^2(-1)^k}{k}\left[\prod_{a=1}^{2k}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_a}{4\pi}\right]\delta(\theta)\left[\binom{2k-2}{k-1}\prod_{j=1}^{2k}\frac{\mathrm{e}^{-rm\cosh\theta_j}}{\cosh\frac{\hat\theta_{jj+1}}{2}}\right.$$
$$\left. -\sum_{j=1}^{k}\sum_{a=1}^{j-1}\sum_{q=\pm}\binom{2k-1}{k-j}(-1)^j\prod_{i=1}^{2k}\frac{\mathrm{e}^{-rm\cosh(\theta_i+q\frac{j-a}{2k}\pi i)}}{\cosh\left(\frac{\hat\theta_{i,i+1}}{2}+q\frac{j-a}{2k}\pi i\right)}\right], \tag{4.13}$$

and $e_{2k+1}(rm) = 0$, where $\theta = \sum_{i=1}^{2k}\theta_i$ and $\hat\theta_{ij} = \theta_i + \theta_j$. Note that for $k = 1$ we exactly recover the result (4.7).

4.2. Boundary entanglement entropy

In order to make use of the form factor expansion (3.7) in the entanglement entropy (2.20), we need to describe in more detail the boundary state introduced in section 2.3. As mentioned there, the state $|B\rangle$ is just a tensor product of boundary states in the individual copies. In integrable models, these have an explicit expression as the famous boundary state introduced by Ghoshal and Zamolodchikov [33]. In the case where no boundary bound state can form, we have

$$|B\rangle = \exp\left(\frac{1}{4\pi}\sum_{\mu=1}^{n}\int_{-\infty}^{\infty}R\left(\frac{\mathrm{i}\pi}{2}-\theta\right)Z_\mu(-\theta)Z_\mu(\theta)\right)|0\rangle. \tag{4.14}$$

The function $R(\theta)$ is the boundary reflection matrix of the integrable QFT (assuming still it has just one particle) and $Z_\mu(\theta)$ are the Faddeev–Zamolodchikov operators, which provide a generalization of the creation-annihilation operators for integrable QFTs with non-trivial interactions [41, 78]. Their main properties are

$$Z_{\mu_1}(\theta_1)\cdots Z_{\mu_k}(\theta_k)|0\rangle = |\theta_1,\ldots,\theta_k\rangle_{\mu_1,\ldots,\mu_k} \quad \text{for} \quad \theta_1 > \cdots > \theta_k$$
$$Z_{\mu_1}(\theta_1)Z_{\mu_2}(\theta_2) = S_{\mu_1\mu_2}(\theta_1-\theta_2)Z_{\mu_2}(\theta_2)Z_{\mu_1}(\theta_1).$$

The tensor-product form of the boundary state indicates that particles living in different copies of the theory do not interact through the presence of the boundary.

Using the boundary operator defined above, we can evaluate all boundary-state overlaps occurring in (3.7) and obtain a large-r expansion for the entanglement entropy from (2.20) in terms of form factors, as for the bulk case. In fact, we could see this expansion simply as coming from expanding in (2.20) the exponential defining the boundary state in (4.14). Either ways, this gives, with the distance between x and the boundary being r,

$$\langle 0|\mathcal{T}(x)|B\rangle = \langle\mathcal{T}\rangle\sum_{k=0}^{\infty}f_{2k}(2rm), \tag{4.15}$$

where

$$\langle \mathcal{T} \rangle f_{2k}(t) = \frac{1}{k!(4\pi)^k} \sum_{j_1,j_2,\ldots,j_k=1}^{n} \left[\prod_{r=1}^{k} \int_{-\infty}^{\infty} \mathrm{d}\theta_r \, \mathrm{e}^{-t\cosh\theta_r} R\left(\frac{\mathrm{i}\pi}{2} - \theta_r\right) \right] \times F_{2k}^{\mathcal{T}|j_1 j_1 j_2 j_2 \cdots j_k j_k}(-\theta_1, \theta_1, \ldots, -\theta_k, \theta_k). \tag{4.16}$$

For example,

$$f_0(t) = 1, \tag{4.17}$$

$$f_2(t) = \frac{n}{4\pi \langle \mathcal{T} \rangle} \int_{-\infty}^{\infty} \mathrm{d}\theta \, R\left(\frac{\mathrm{i}\pi}{2} - \theta\right) F_2^{\mathcal{T}|11}(-2\theta) \, \mathrm{e}^{-t\cosh\theta}, \tag{4.18}$$

and so on. Therefore, we can write the boundary entanglement entropy as

$$S_A^{\text{boundary}}(rm) = -\frac{c}{6}\log(\varepsilon m) + \frac{U^{\text{model}}}{2} + \sum_{k=1}^{\infty} s_{2k}(2rm), \tag{4.19}$$

with

$$s_{2k}(2rm) = -\frac{\mathrm{d} f_{2k}(2rm)}{\mathrm{d}n}\bigg|_{n=1}. \tag{4.20}$$

Here, we choose the universal model-dependent constant U^{model} to be the same as in the bulk expansion (4.1). This, as we explained in sections 1 and 2.4, completely fixes the UV behaviour of the entropy in the boundary cases, and in particular fixes the relation between the non-universal short-distance cutoff ε and the correlation length.

4.2.1. Next-to-leading order IR correction to the boundary entanglement entropy. The series (4.19) is of immediate use to provide a large-distance expansion of the boundary entanglement entropy. In particular, we may easily obtain the general form of the first correction to saturation at large distances. From (4.18) and (4.20) we see that the next-to-leading order correction for r large is given by

$$\begin{aligned} s_2(2rm) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \, R\left(\frac{\mathrm{i}\pi}{2} - \theta\right) \left[\frac{\mathrm{d} F_2^{\mathcal{T}|11}(-2\theta)}{\mathrm{d}n} \right]_{n=1} \mathrm{e}^{-2rm\cosh\theta} \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}\theta \, \frac{R\left(\frac{\mathrm{i}\pi}{2} - \theta\right)}{\left(\cosh\frac{\theta}{2}\right)^2} \left[\frac{F_{\min}^{\mathcal{T}|11}(-2\theta)}{F_{\min}^{\mathcal{T}|11}(\mathrm{i}\pi)} \right]_{n=1} \mathrm{e}^{-2rm\cosh\theta}. \end{aligned} \tag{4.21}$$

This correction is much more model-dependent than the similar correction in the bulk case, both through the boundary reflection matrix and through the minimal form factors (which come from the scattering matrix); there is in fact little to say concerning its general features.

In order to study the form of $s_2(2mr)$ in more detail and investigate higher order corrections, we now resort to the particular example of the Ising model.

4.2.2. Boundary entanglement entropy of the Ising model. Expansion (4.19) is not only an efficient large-distance expansion, but also an exact representation. Hence, it can be used to analyse the short-distance behaviour of the boundary entanglement entropy as well, by resumming all the terms of the series. In what follows, we will reproduce the main arguments and results of such a study of in the case of the Ising model, done in [32].

Let us recall the types of integrable boundary conditions that have been found for the Ising model, whose two-particle S-matrix is simply -1. A family that was studied in much

detail in [33] is that corresponding to the presence of a magnetic field that couples to the Ising spin field on the boundary. The spin field is the order parameter, hence we are looking at the scaling limit of the Ising spin chain in a transverse magnetic field whose magnitude is slightly below its critical value (ordered regime), and with a parallel magnetic field on the boundary. The corresponding boundary reflection matrix is given by

$$R(\theta) = -\mathrm{i}\tanh\frac{1}{2}\left(\theta - \frac{\mathrm{i}\pi}{2}\right)\frac{\kappa - \mathrm{i}\sinh\theta}{\kappa + \mathrm{i}\sinh\theta}, \tag{4.22}$$

which includes, for special values of the parameter κ, the following physically different types of integrable boundary conditions:

- *free boundary condition*: $\kappa = 1$;
- *fixed boundary condition*: $\kappa = -\infty$;
- *magnetic boundary conditions* (interpolating between the previous two): $\kappa = 1 - \frac{h^2}{2m}$, where h is a boundary magnetic field $0 < h < \infty$. The free boundary condition would then correspond to $h = 0$ whereas the fixed boundary condition is equivalent to having a infinitely large magnetic field fixed at the boundary.

Boundary corrections to the expectation values of the energy and disorder field in the Ising theory were computed using this reflection matrix in [79].

In the cases where $\kappa > 0$, the reflection matrix has a pole on the imaginary line on the physical sheet, $0 < \mathrm{Im}(\theta) \leqslant \mathrm{i}\pi/2$. This implies that the boundary state expression (4.14) is not correct. A modified expression exists [33], but for simplicity, we will not analyse this case here. Hence, throughout we will consider $\kappa \leqslant 0$. Note that the case $\kappa = 0$ does not require modifications, since the residue of the R-matrix vanishes at this point. At $\kappa = 0$, the bound state becomes weakly bound and propagates far into the bulk.

The cases $\kappa > -1$, i.e. $h < h_c = 2\sqrt{m}$, are also somewhat special. In these cases, the R-matrix still has a pole on the imaginary θ line, although not on the physical strip when $\kappa \leqslant 0$. As noted in [33], the case $\kappa = -1$ corresponds to a 'critical' value of the magnetic field, h_c, at which the reflection matrix happens to have a third order zero at $\theta = 0$.

We note also that the R matrix at $\kappa = 0$ is just equal to the negative of the fixed-boundary condition R matrix, $\kappa = -\infty$, and that

$$R\left(\frac{\mathrm{i}\pi}{2} - \theta\right) = -R\left(\frac{\mathrm{i}\pi}{2} + \theta\right). \tag{4.23}$$

In order to make it clear that we are dealing with the Ising model, we will explicitly write the κ dependence in the boundary corrections, $s_{2k}(t) \mapsto s_{2k}(t, \kappa)$.

Reproducing the main arguments of [32], we will analyse the corrections $s_{2k}(t, \kappa)$ order by order. This will provide, by resummation of these corrections, an exact evaluation of the constant $V(\kappa)$ appearing in (1.11).

As was mentioned in section 2.4, we expect the constant $V(\kappa)$ to depend only on the conformal boundary condition that is reached as we bring the length r to zero (i.e. in the limit $r \ll m^{-1}$). There are two conformal boundary conditions for the Ising model: zero magnetic field, i.e. free boundary condition, and infinite magnetic field, i.e. fixed boundary condition (although, to be precise, the latter one has two possibilities, for the two orientations of the boundary magnetic field). The first one is a UV boundary condition, and the second an IR boundary condition, from the point of view of the boundary renormalization group flow. Hence, for any fixed κ, we expect that in the UV limit $mr \to 0$, we obtain a situation described by the UV, free boundary condition. On the other hand, if the system is already at a fixed boundary condition $\kappa = -\infty$, we expect that as $mr \to 0$ we obtain a situation described by the IR, fixed boundary condition. Hence, $V(\kappa)$ should be the same for any finite κ, but should take another value for $\kappa = -\infty$.

We may wish to obtain the boundary entanglement entropy in the situation where the bulk is conformal, but the boundary itself is not conformal, with a magnetic boundary condition, that depends on the magnetic field. In this case, we have to send the bulk correlation length m^{-1} to infinity while keeping both r and the magnetic, boundary correlation length η finite, and of the same order. The result will be a universal function of the ratio r/η. This is the limit where m^{-1} is greater than all other scales, so that we must take simultaneously $mr \to 0$, $\kappa \to -\infty$ in an appropriate way. The way this limit is taken can be understood through CFT considerations, but the four-particle calculation below will make it clear that it is the product κmr that we need to keep fixed. In terms of the magnetic field, this means that we need to keep the product $h^2 r$ fixed (both ways are equivalent in the limit $\kappa \to -\infty$). That is, the boundary correlation length η is of the order of h^{-2}. We will not investigate this situation much further, although we will discuss it below in relation with the four-particle contribution. But we note that after taking this particular limit, if we then make the boundary correlation length η very large or very small compared to r, we obtain a free/fixed conformal boundary condition. From this point of view, it is clear that it is indeed the point $\kappa = -\infty$ that divides the regions in κ where the UV/IR conformal boundary condition is reached as we take $mr \to 0$ keeping κ fixed.

4.2.3. Two-particle correction to the entropy in the Ising model. In terms of the R-matrix (4.22) and the minimal form factor (3.34), the correction (4.21) becomes

$$s_2(t,\kappa) = -\frac{1}{8}\int_{-\infty}^{\infty} \mathrm{d}\theta \left(\frac{\kappa + \cosh\theta}{\kappa - \cosh\theta}\right)\left(\frac{\cosh\theta - 1}{\cosh^2\theta}\right) \mathrm{e}^{-t\cosh\theta}. \tag{4.24}$$

We see that the correction $s_2(t,\kappa)$ is finite for all values of t, including $t = 0$ (zero distance). In particular, this correction is not expected to contribute to the logarithmic term in the small-mr behaviour of the boundary entanglement entropy (1.11). At $t = 0$ it is possible to evaluate the integral above explicitly:

$$c_2(\kappa) := s_2(0,\kappa) = \frac{1}{4} - \frac{\pi}{8} + \frac{\pi}{4\kappa} - \frac{\sqrt{1-\kappa}\,(\pi + 2\arcsin(\kappa))}{4\kappa\sqrt{1+\kappa}}, \tag{4.25}$$

and in particular

$$c_2(-1) = \frac{10 - 3\pi}{8} \qquad \text{and} \qquad c_2(0) = \frac{\pi - 2}{8}. \tag{4.26}$$

The number $c(\kappa)$ will contribute to the constant $V(\kappa)$ of (1.11).

4.2.4. Four-particle correction to the entropy in the Ising model. It is interesting to analyse in some detail the next correction to the boundary entanglement entropy for the Ising model. For boundary theories, this is the first correction for which the issue of finding the right analytic continuation in n plays again a crucial role. As it turns out that crucial role extends to all other higher order corrections, which were explicitly found for the Ising model [32]. Also, the contribution $s_4(t,\kappa)$ gives us a lot of insight as to how the short-distance behaviour in (1.11) can be recovered; in particular, how the two values of $V(\kappa)$ can come out, and how they are connected by the massless, non-conformal magnetic boundary situation explained in section 4.2.2.

The four-particle boundary correction is given by

$$s_4(t,\kappa) = -\frac{1}{2}\left[\prod_{k=1}^{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_k}{4\pi} R\left(\frac{\mathrm{i}\pi}{2} - \theta_k\right)\mathrm{e}^{-t\cosh\theta_k}\right]\frac{\mathrm{d}}{\mathrm{d}n} \times \left[\sum_{i,j=1}^{n}\frac{1}{\langle\mathcal{T}\rangle}F_4^{\mathcal{T}|iijj}(-\theta_1,\theta_1,-\theta_2,\theta_2)\right]_{n=1}, \tag{4.27}$$

where

$$\sum_{i,j=1}^{n} F_4^{\mathcal{T}|iijj}(-\theta_1,\theta_1,-\theta_2,\theta_2) = n\sum_{j=0}^{n-1} F_4^{\mathcal{T}|1111}\big(-\theta_1,\theta_1,(-\theta_2)^j,\theta_2^j\big). \tag{4.28}$$

Here we have used (3.39) (in the case of the Ising model, there is no need for the ordering of the particle indices), as well as the notation $\theta^j = \theta + 2\pi \mathrm{i} j$. Employing (3.40) we find

$$\frac{n}{\langle\mathcal{T}\rangle}\sum_{j=0}^{n-1} F_4^{\mathcal{T}|1111}\big(-\theta_1,\theta_1,(-\theta_2)^j,\theta_2^j\big)$$
$$= n^2 K(2\theta_1)K(2\theta_2) + n\sum_{j=0}^{n-1}\big(K\big(\theta_{12}^{-j}\big)K\big(\theta_{12}^{j}\big) - K\big(\hat{\theta}_{12}^{-j}\big)K\big(\hat{\theta}_{12}^{j}\big)\big), \tag{4.29}$$

where the function $K(x)$ was introduced in (3.38), and we use the 'hat' notation introduced after equation (4.13). It is simple to show that the $n^2K(2\theta_1)K(2\theta_2)$ term will give no contribution to the derivative at $n = 1$ so that only the terms in the sum remain. These terms will give a contribution, since, employing (3.43) and (3.44), they can actually be rewritten as

$$-n\sum_{j=0}^{n-1}[|K(\theta_{12}-2\pi\mathrm{i}j)|^2 - |K(\hat{\theta}_{12}-2\pi\mathrm{i}j)|^2] \stackrel{\int}{=} -2n\sum_{j=0}^{n-1}[|K(\theta_{12}-2\pi\mathrm{i}j)|^2], \tag{4.30}$$

where the symbol $\stackrel{\int}{=}$ means equality up to integration in θ_1 and θ_2 inside the integral (4.27). We can then use the result (4.11) to show that the sum (4.30) is proportional to $\delta(\theta_{12})$. The four-particle correction to the saturation value of the Ising entanglement entropy is therefore

$$s_4(t,\kappa) = \frac{1}{32}\int_{-\infty}^{\infty} \mathrm{d}\theta\, R\left(\frac{\mathrm{i}\pi}{2}-\theta\right)^2 \mathrm{e}^{-2t\cosh\theta}$$
$$= \frac{1}{32}\int_{-\infty}^{\infty} \mathrm{d}\theta \left(\frac{\kappa+\cosh\theta}{\kappa-\cosh\theta}\right)^2 \frac{1-\cosh\theta}{1+\cosh\theta}\,\mathrm{e}^{-2t\cosh\theta}.$$

The four-particle contribution is divergent as $t \to 0$. This points to the fact that $s_4(t,\kappa)$ will contribute to the logarithmic term in the short-distance behaviour in (1.11). Technically, the reason for this is that the integrand of (4.31) is a function that tends to the value -1 as $\theta \to \infty$ when $t = 0$. Therefore the integral at $t = 0$ is divergent. In order to find the precise behaviour of the correction as t approaches 0, one can rewrite the integral above as

$$s_4(t,\kappa) = \frac{1}{32}\int_{-\infty}^{\infty} \mathrm{d}\theta \left[\left(\frac{\kappa+\cosh\theta}{\kappa-\cosh\theta}\right)^2 \frac{1-\cosh\theta}{1+\cosh\theta}+1\right]\mathrm{e}^{-2t\cosh\theta} - \frac{1}{16}K_0(2t). \tag{4.31}$$

The behaviour of the Bessel function as t goes to zero is well known,

$$K_0(2t) = -\gamma - \log(t) + O(t^2\log t), \tag{4.32}$$

where $\gamma = 0.577\,216\ldots$ is the Euler–Mascheroni constant. Written in this form, the integral part is now a finite constant at $t = 0$, and we may define

$$c_4(\kappa) = \frac{1}{16}\int_0^{\infty} \mathrm{d}\theta \left[\left(\frac{\kappa+\cosh\theta}{\kappa-\cosh\theta}\right)^2 \frac{1-\cosh\theta}{1+\cosh\theta}+1\right]$$
$$= -\frac{1}{8(1+\kappa)^2}\left[2\kappa - 3\kappa^2 - 1 + \frac{\kappa\,(2\kappa-1)\,(\pi+2\arcsin\kappa)}{\sqrt{1-\kappa^2}}\right], \tag{4.33}$$

with in particular

$$c_4(-1) = \tfrac{23}{120}. \tag{4.34}$$

Therefore we have

$$s_4(t,\kappa) = \frac{1}{16}\log(t) + \frac{\gamma}{16} + c_4(\kappa) + o(1). \tag{4.35}$$

In view of the short-distance behaviour of the entanglement entropy (1.11), we expect that the coefficients of the logarithmic divergencies at small rm will add up to the finite number $1/12$ when all corrections are considered. This has been proven numerically in [32]. Also, the constant $c_4(\kappa)$, like $c_2(\kappa)$ above, is a part of the constant $V(\kappa)$ in (1.11); again, in principle one should add up all such constants, for all corrections, in order to obtain $V(\kappa)$.

Recall that $V(\kappa)$ takes only two possible values, one for $\kappa > -\infty$ and the other for $\kappa = -\infty$. This implies that $V(-\infty) \neq \lim_{\kappa\to-\infty} V(\kappa)$. It turns out that this inequality is true for $c_4(\kappa)$, and should hold as well as for infinitely many constants $c_{2\ell}(\kappa)$ [32]. Technically, we observe that although the integral in (4.31) has the same value for $\kappa = -\infty$ as for $\kappa = 0$ for any $t > 0$, we have $\lim_{\kappa\to-\infty} c_4(\kappa) = 3/8$, different from $c_4(0) = 1/8$. The explanation is that the limit $t \to 0$ of the integral in (4.31) as a function of κ is not uniform. For all values of $t > 0$ we have $s_4(t, -\infty) = s_4(t, 0)$, and there is a maximum for $\kappa \in (-\infty, 1)$ at a unique value $\kappa = \kappa_0$. But as t becomes smaller, the position of this maximum shifts towards more negative values, until it reaches $-\infty$ at $t = 0$. There, if we take away the constant (as function of κ) term $\frac{1}{16}\log(t)$ in order to make the limit finite, the value of the maximum itself reaches $\lim_{\kappa\to-\infty} c_4(\kappa)$. It is also possible to observe in the integral in (4.33) that the symmetry between $\kappa = -\infty$ and $\kappa = 0$ is broken. Indeed, if $\kappa \to -\infty$, the term in parentheses can be approximated by 1 except for values of θ where $\kappa + \cosh\theta \approx 0$. These are very large values of θ, but they are not damped by any other factor, hence the mistake in approximating by 1 is non-negligible for any κ.

This means that expansion (4.35) is valid only for $\kappa > -\infty$. For the case $\kappa = -\infty$, that is, the fixed boundary condition, we have to consider the other order of the limits: first $\kappa \to -\infty$, then $t \to 0$. By the symmetry between $\kappa = -\infty$ and $\kappa = 0$, we define

$$c_4(-\infty) =: c_4(0) = \tfrac{1}{8} \tag{4.36}$$

so that (4.35) still holds in the case of a fixed boundary condition, $\kappa = -\infty$.

Recall our discussion in section 4.2.2, about the particular massless limit whereby the boundary is still magnetic, not conformal. In this perspective, it is instructive to obtain a more general small-t expansion, where we take simultaneously $\kappa \to -\infty$. Let us consider $t \to 0$ with $-\kappa t = a$ fixed. We may use the change of variable $s = \cosh\theta - 1$ and write $s_4(t,\kappa)$ as

$$\frac{1}{16}\int_0^\infty \mathrm{d}s \left(\frac{k+1+s}{k-1-s}\right)^2 \left(-\frac{\sqrt{s}}{(s+2)^{3/2}} + \frac{1}{s+1}\right) \mathrm{e}^{-2t(s+1)} - \frac{1}{16}\int_0^\infty \mathrm{d}s \left(\frac{k+1+s}{k-1-s}\right)^2 \frac{\mathrm{e}^{-2t(s+1)}}{s+1}. \tag{4.37}$$

The first integral as a function of κ has a uniform limit as $t \to 0$ on $\kappa \in [-\infty, 0)$, so that we can directly take $\kappa = -\infty$ and $t = 0$; this gives $(2 - \log 2)/16$. The second integral does not have a uniform limit, but it can be evaluated explicitly:

$$\frac{1}{16(a+t)}(4a\,\mathrm{e}^{-2t} - (a+t)\Gamma(0,2t)\,\mathrm{e}^{2t} - 8a(a+t)\,\mathrm{e}^{2(a+t)}\Gamma(0,2(a+t))),$$

where $\Gamma(z,u)$ is the incomplete gamma function, $\int_u^\infty v^{z-1}\,\mathrm{e}^{-v}\,\mathrm{d}v$. The small-$t$ limit can then easily be taken:

$$s_4(t,-a/t) = \frac{1}{16}\log(t) + \frac{\gamma}{16} + c_4^\natural(a) + O(t), \tag{4.38}$$

where

$$c_4^\natural(a) = \tfrac{3}{8} - \tfrac{1}{2}a\,\mathrm{e}^{2a}\Gamma(0, 2a). \tag{4.39}$$

It is easy to see that $c_4^\natural(a)$ interpolates between $\lim_{\kappa\to-\infty} c_4(\kappa)$ at $a = 0$ to $c_4(-\infty)$ at $a = \infty$.

4.2.5. Higher order corrections to the entropy in the Ising model. Many of the subtleties observed in the previous section for the four-particle correction to the boundary entanglement entropy generalize to higher orders, in particular the need to find the analytic continuation in n of our expressions and the non-commutativity of the limits observed above. We do not wish to go into the details of the computations involved, which are very technical and cumbersome, even for the Ising model. However, let us give an indication of how it works.

The main complication arises from the fact that the pole- and zero-structure of the form factors (in particular, the way some poles and zeroes cancel each other) changes substantially as soon as n is allowed to take non-integer values. As a consequence, the phenomenon observed in [31], and recalled above, of non-uniform convergence of form factors as $n \to 1$ is generalized to a non-uniform convergence as $n \to k'$ for all positive integers $k' \leqslant k$ for the $2k$-particle form factor. In order to obtain the correct analytic continuation, the principle is that it is obtained from the analytic function that describes form factor contributions at values of n large enough. This amounts to evaluating first the contribution we would obtain analytically continuing in n around some integer and then adding the residues of all the extra poles that are crossed by the integration contours when bringing n from infinity. This is naturally not the only analytic continuation that is possible and therefore it becomes quite crucial to find ways of checking it for consistency. The exactly UV behaviour shown below provides good support, see [32] for a more extensive discussion.

The general formula for all higher particle corrections found in [32] takes the form:

$$\begin{aligned} s_{4k}(t,\kappa) = \frac{\pi^2(-1)^k}{2k}\left[\prod_{a=1}^{2k}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_a}{4\pi}\right]\delta(\theta)\left[\binom{2k-2}{k-1}\prod_{j=1}^{2k}\frac{\mathrm{e}^{-t\cosh\theta_j}R\left(\frac{\mathrm{i}\pi}{2}-\theta_j\right)}{\cosh\frac{\hat\theta_{jj+1}}{2}}\right.\\ \left. -\sum_{j=1}^{k}\sum_{a=1}^{j-1}\sum_{q=\pm}\binom{2k-1}{k-j}(-1)^j\prod_{i=1}^{2k}\frac{\mathrm{e}^{-t\cosh(\theta_i+q\frac{j-a}{2k}\pi\mathrm{i})}R\left(\frac{\mathrm{i}\pi}{2}-(\theta_i+q\frac{j-a}{2k}\pi\mathrm{i})\right)}{\cosh\left(\frac{\hat\theta_{i,i+1}}{2}+q\frac{j-a}{2k}\pi\mathrm{i}\right)}\right] \end{aligned} \tag{4.40}$$

and similarly

$$\begin{aligned} s_{4k+2}(t,\kappa) = \frac{\pi^2(-1)^k}{2k+1}\left[\prod_{a=1}^{2k+1}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_a}{4\pi}\right]\delta(\theta)\sum_{j=1}^{k}\sum_{a=1}^{j}\sum_{q=\pm}\binom{2k}{k-j}(-1)^j q\\ \times\prod_{i=1}^{2k+1}\frac{\mathrm{e}^{-t\cosh(\theta_i+q\frac{j-a+1/2}{2k+1}\pi\mathrm{i})}R\left(\frac{\mathrm{i}\pi}{2}-(\theta_i+q\frac{j-a+1/2}{2k+1}\pi\mathrm{i})\right)}{\cosh\left(\frac{\hat\theta_{i,i+1}}{2}+q\frac{j-a+1/2}{2k+1}\pi\mathrm{i}\right)}, \end{aligned} \tag{4.41}$$

both formulae hold for $k = 1, 2, 3, \ldots$. All integrals involved are absolutely convergent. Note that for $k = 1$ in (4.40) we recover the result (4.31) as it should be. Setting $k = 0$ in (4.41) does not give (4.24), since $k = 0$ is out of the range of applicability of these formulae; the two-particle case $s_2(t, \kappa)$ is a special case.

From these expressions it is also possible to extract the divergent part as $t \to 0$, as done for the four-particle contribution in the previous paragraph. This is achieved by simply setting $t = 0$ in all expressions, after subtracting 1 from the product of R-matrices. We then obtain $s_{4k}(t, \kappa) \sim c_{4k}(\kappa) + e_{2k}(t)$ and $s_{4k+2}(t, \kappa) \sim c_{4k+2}$ as $t \to 0$, for all $k = 1, 2, 3, \ldots$, where

$e_{2k}(t)$ are the expansion coefficients (4.13) in the expression for the bulk entanglement entropy (see (4.1)), and where

$$c_{4k}(\kappa) = \frac{\pi^2(-1)^k}{2k}\left[\prod_{a=1}^{2k}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_a}{4\pi}\right]\delta(\theta)\left[\binom{2k-2}{k-1}\frac{\prod_{j=1}^{2k}R\left(\frac{\mathrm{i}\pi}{2}-\theta_j\right)-1}{\prod_{j=1}^{2k}\cosh\frac{\hat{\theta}_{jj+1}}{2}}\right.$$
$$\left.-\sum_{j=1}^{k}\sum_{a=1}^{j-1}\sum_{q=\pm}\binom{2k-1}{k-j}(-1)^j\frac{\prod_{i=1}^{2k}R\left(\frac{\mathrm{i}\pi}{2}-(\theta_i+q\frac{j-a}{2k}\pi\mathrm{i})\right)-1}{\prod_{i=1}^{2k}\cosh\left(\frac{\hat{\theta}_{i,i+1}}{2}+q\frac{j-a}{2k}\pi\mathrm{i}\right)}\right] \tag{4.42}$$

and similarly

$$c_{4k+2}(\kappa) = \frac{\pi^2(-1)^k}{2k+1}\left[\prod_{a=1}^{2k+1}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_a}{4\pi}\right]\delta(\theta)\sum_{j=1}^{k}\sum_{a=1}^{j}\sum_{q=\pm}\binom{2k}{k-j}(-1)^j q$$
$$\times\prod_{i=1}^{2k+1}\frac{R\left(\frac{\mathrm{i}\pi}{2}-(\theta_i+q\frac{j-a+1/2}{2k+1}\pi\mathrm{i})\right)}{\cosh\left(\frac{\hat{\theta}_{i,i+1}}{2}+q\frac{j-a+1/2}{2k+1}\pi\mathrm{i}\right)}, \tag{4.43}$$

both formulae for $k = 1, 2, 3, \ldots$. The constant $V(\kappa)$ that characterizes the UV behaviour of the boundary entanglement entropy as shown in (1.11) can now be obtained as follows:

$$S_A^{\text{boundary}}(rm) = \frac{1}{12}\log(m\varepsilon) + \frac{U^{\text{Ising}}}{2} + \sum_{k=1}^{\infty}s_{2k}(2rm,\kappa)$$
$$\sim \frac{1}{12}\log(m\varepsilon) + \frac{U^{\text{Ising}}}{2} + \sum_{k=1}^{\infty}c_{2k}(\kappa) + \frac{1}{2}\left(S_A^{\text{bulk}}(2rm) - U^{\text{Ising}} - \frac{1}{6}\log(m\varepsilon)\right)$$
$$\sim -\frac{1}{12}\log(2r/\varepsilon) + \sum_{k=1}^{\infty}c_{2k}(\kappa), \tag{4.44}$$

hence

$$V(\kappa) = \sum_{k=1}^{\infty}c_{2k}(\kappa). \tag{4.45}$$

Naturally, in order for (4.45) to be a correct representation of $V(\kappa)$, the infinite sum over k should give a finite result. This is quite subtle, as form factor expansions are expected to provide convergent series expansion for finite distances, but not necessarily at zero distance. In [32], exact evaluations of the first few coefficients $c_{2k}(\kappa)$ for $\kappa = 0, -1, -\infty$, and extrapolation to higher k, gave strong indications that the series is indeed convergent. More precisely we have managed to obtain closed formulae for $c_{2k}(0)$ and $c_{2k}(-\infty)$:

$$c_{4k}(0) = \frac{1}{8k(2k-1)}, \qquad c_{4k+2}(0) = \frac{\pi}{2^{4k+1}}\binom{2k-1}{k-1}^2 - \frac{1}{4(2k+1)}, \tag{4.46}$$

and $c_{4k}(-\infty) = c_{4k}(0)$ and $c_{4k+2}(-\infty) = -c_{4k+2}(0)$. This then gives

$$V(0) = \frac{\pi-2}{8} + \sum_{k=1}^{\infty}c_{4k}(0) = \log\sqrt{2} \tag{4.47}$$

and

$$V(-\infty) = \frac{2-\pi}{8} + \sum_{k=1}^{\infty}(c_{4k}(0) - c_{4k+2}(0)) = 0. \tag{4.48}$$

We have also computed the values of $c_{2k}(-1)$ for k up to 4 and found that $V(-1) \simeq 0.321\,966\ldots$. This value is compatible with $\log\sqrt{2} = 0.346\,574\ldots$. Therefore, as expected, in the UV limit the entropy only depends on whether the magnetic field h is finite (free boundary conditions) or infinite (fixed boundary conditions). We can then summarize our result as

$$V(\kappa) = \begin{cases} \log\sqrt{2} & (\kappa > -\infty) \\ 0 & (\kappa = -\infty). \end{cases} \tag{4.49}$$

4.2.6. Connection with the boundary entropy. It is natural to interpret $V(\kappa)$ as a 'boundary entanglement': the contribution of the boundary to the entanglement between the region A and the rest. Naturally, for fixed boundary condition, there should be no contribution at all, since the boundary does not experience quantum fluctuations. Our result (4.49) shows that we have chosen the correct large-distance normalization to have $V(-\infty) = 0$. On the other hand, for free boundary conditions, the boundary fluctuates and should participate to the entanglement. This is in agreement with $V(\kappa > -\infty) = \log\sqrt{2} > 0$.

In fact, it was shown in [32] that we may connect $V(\kappa)$ to the *boundary entropy* s, a quantity that essentially counts the number of degrees of freedom pertaining to a boundary. This quantity is simply given by $s = \log g$ where g is the boundary degeneracy introduced by Affleck and Ludwig [54]. In particular for a bulk CFT, they showed that $g = \langle 0|\tilde{B}\rangle$ where $|0\rangle$ is the bulk CFT ground state, and $|\tilde{B}\rangle$ is a boundary state of norm 1 in the bulk CFT Hilbert space (in particular, $s \leqslant 0$). It is considering the boundary entropy that Friedan and Konechny [80] were able to provide a proof of the 'g-theorem': that the g-function decreases in the RG flow from UV to IR. For the Ising model, it turns out that $s = 0$ in the free boundary case and $s = -\sqrt{2}$ in the fixed boundary case.

In order to understand the general relation between $V(\kappa)$ and s, we must make use of the formula derived by Calabrese and Cardy [24] connecting the boundary entropy to the entanglement entropy at criticality:

$$s = S_A^{\text{boundary}}(r)_{\text{critical}} - \tfrac{1}{2} S_A^{\text{bulk}}(2r)_{\text{critical}}. \tag{4.50}$$

The result is independent of r, since both entanglement entropies have the same logarithmic r-dependence. Naturally, the universal entanglement entropy, as we emphasized in section 1, must be defined by subtracting a non-universal constant, the logarithm of the correlation length. Hence, the universal entanglement entropy is only defined up to addition of a r-independent and boundary-independent constant. We provided in section 1 precise ways of fixing this constant in the bulk and boundary cases, by looking at asymptotics, with a massive bulk. However, formula (4.50) holds if and only if both entanglement entropies are evaluated in the same cut-off scheme. For instance, both should be evaluated on the same lattice at infinite correlation length, and taking the distance r to be the same number of lattice sites (this number tending to infinity). Formula (4.50) is not of immediate use in the context of QFT, because the requirement of a consistent cut-off scheme is not a universal requirement.

It turns out that the use of massive QFT solves this problem. More precisely, we can make sure that we have the same cut-off schemes in calculating $S_A^{\text{boundary}}(r)_{\text{critical}}$ and $S_A^{\text{bulk}}(2r)_{\text{critical}}$ by connecting them to our universal definitions of the entanglement entropy in the bulk and boundary massive cases. Recall that our definitions involve the inherently massive large-distance asymptotics. More precisely, we can show that, in the Ising model,

$$V(\kappa) = S_A^{\text{boundary}}(r)_{\text{critical}} - \tfrac{1}{2} S_A^{\text{bulk}}(2r)_{\text{critical}} + \log\sqrt{2}. \tag{4.51}$$

The entanglement entropies should be evaluated at the UV fixed point if κ is finite and at the IR fixed point if $\kappa = -\infty$.

This implies that

$$V(\kappa) = s - \log\sqrt{2}. \tag{4.52}$$

This is in agreement with numerical results obtained in [81]. In fact, it was argued in [32] that in general the shift $\log\sqrt{2}$ should be replaced by $-\log\mathcal{C}$, where $\mathcal{C}^2$ is the fraction of the ground-state degeneracy broken by the boundary condition for large enough h (at the IR point). In the Ising model, the ground state has a double degeneracy (the two orientations of the spin), and a large magnetic field h breaks it to 1, so that $\mathcal{C}^2 = 1/2$. This implies

$$V(\kappa) = s - \log\mathcal{C}. \tag{4.53}$$

Such an 'extra' contribution $-\log\mathcal{C}$ to $s = \log g$ was also found in [82] in calculations using thermodynamic Bethe ansatz techniques (although it is important to note that the thermodynamic Bethe ansatz used there is fundamentally different from our approach based on form factors). In [82] the flow of g between critical points was studied for several families of minimal Toda field theories with ground-state degeneracy k and with a boundary completely breaking it. The factor $\mathcal{C} = 1/\sqrt{k}$ was termed 'symmetry factor'. Interestingly, it was shown that at the infrared point of these models, one gets $g = \mathcal{C}$. Hence, for massive models with spontaneously broken order-parameter symmetry, and with an order-parameter boundary perturbation, one should find $V(\kappa) = s - s_{\rm IR} \geqslant 0$, where $s_{\rm IR} = \log\mathcal{C}$ is the infrared value of s. For the same order-parameter perturbation both on the bulk and boundary, one should simply find $V(\kappa) = s \leqslant 0$. All these considerations should not depend on integrability.

Let us now discuss briefly how we obtain (4.51), following [32]. Consider the entanglement entropy $S_A^{\rm boundary}(x_1, x_2)$ defined in section 2.3, but with a slightly different normalization specified below; we will denote it $\tilde{S}_A(x_1, x_2)$. We may uniquely fix the cutoff, for instance by requiring the conformal normalization, $\tilde{S}_A(x_1, x_2) \sim \frac{c}{3}\log(|x_2 - x_1|/\varepsilon) + o(1)$ as $x_2 \to x_1$. But this short distance behaviour is the bulk critical entanglement entropy, so that we have chosen ε with $S_A^{\rm bulk}(r)_{\rm critical} = \frac{c}{3}\log(r/\varepsilon)$. Using the same object, hence with the same lattice spacing, we can also define the critical entanglement entropy in the boundary case, $S_A^{\rm boundary}(r)_{\rm critical}$, following section 2.3. Note first that for both x_2 and x_1 far from the boundary and far from each other, the entanglement entropy saturates to some constant $-\frac{c}{3}\log(m\varepsilon) + \tilde{U}$ thanks to the presence of the mass. This saturation is a sum of the contributions of the two boundary points at x_1 and x_2, each one contributing $-\frac{c}{6}\log(m\varepsilon) + \tilde{U}/2$. Now for x_2 alone being far from the boundary, we get our usual boundary case, but with an extra contribution of the boundary point x_2 at infinity. Subtracting this contribution, we find $\tilde{S}_A(r)$ with r the distance between x_1 and the boundary: this is our usual boundary case, with possibly a different cut-off definition. Taking the limit $rm \to 0$, we get $\tilde{S}_A(r) \sim S_A^{\rm boundary}(r)_{\rm critical} + o(1)$.

We must now relate $\tilde{S}_A(r)$ to $S_A^{\rm boundary}(r)$. We know that for r large, the former saturates to $-\frac{c}{6}\log(m\varepsilon) + \tilde{U}/2$, and the latter to $-\frac{c}{6}\log(m\varepsilon) + U/2$. The difference between $\tilde{U}$ and U is that the former really just counts the contributions to the entanglement entropy around the boundary points of A when A is large, whereas the latter counts an additional entropy due to the ground-state degeneracy. Indeed, the constant U occurs in the bulk, which should be seen as a large-volume limit of a periodic space where both spin-up $|0, +\rangle$ and spin-down $|0, -\rangle$ ground states are counted. The constant $\tilde{U}$ occurs when the system has a boundary, where the ground state is fixed if the magnetic field is large enough. Hence we have

$$U = \tilde{U} + \log 2. \tag{4.54}$$

Then we find $S_A^{\rm boundary}(r)_{\rm critical} = \frac{c}{3}\log(r/\varepsilon) + V(\kappa) - \log\sqrt{2}$, which shows (4.51).

There is a nice technical way of obtaining (4.54), using branch-point twist fields [32]. Allowing fluctuations amongst the two degenerate ground states corresponds to choosing

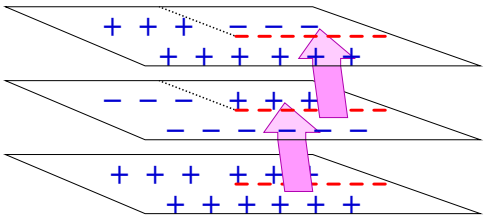

Figure 9. How a vacuum of the form $\cdots \otimes |0,+\rangle \otimes |0,-\rangle \otimes |0,+\rangle \otimes \cdots$ becomes a two-particle state after going through a branch-point twist field.

$|0\rangle = ((|0,+\rangle + |0,-\rangle)/\sqrt{2})^{\otimes n}$ in the n-copy model. Our branch-point twist field form factors assume the use of this ground state. This can be seen by recalling that particles in the Ising model are domain walls, separations between up- and down-spin domains. In order to have a non-zero matrix element of the branch-point twist field between a state with a particle on one sheet and another particle on another sheet, and the n-copy ground state, there must be components of the ground state where one sheet is on $|0,+\rangle$ and another sheet is on $|0,-\rangle$. See figure 9. A completely up ground state $|0,+\rangle^{\otimes n}$, for instance, would not admit such form factors. Since all configurations of an even number of particles distributed amongst the n sheets are allowed, we must use the completely symmetric ground state. Then, the constant $\tilde{U}$ is the large-distance limit of the bulk entanglement entropy evaluated from $\langle 0,+|^{\otimes n}\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)|0,+\rangle^{\otimes n}$, whereas U is that obtained from $\langle 0|\tilde{\mathcal{T}}(r_1)\mathcal{T}(r_2)|0\rangle$ with the symmetric ground state. The difference can be computed explicitly. Note that matrix elements

$$\langle 0,\epsilon_1'| \otimes \cdots \otimes \langle 0,\epsilon_n'|\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)|0,\epsilon_1\rangle \otimes \cdots \otimes |0,\epsilon_n\rangle$$

have zero large-distance limit unless $\epsilon_i = \epsilon_j = \epsilon_j'$ for all i, j (that is, all signs are the same), since otherwise domain walls will have to propagate between the twist fields. Hence we immediately find

$$\langle 0|\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)|0\rangle \sim 2^{1-n}\langle 0,+|^{\otimes n}\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)|0,+\rangle^{\otimes n} \tag{4.55}$$

at large distances, so that, taking derivatives with respect to n, we have

$$-\left(\frac{\mathrm{d}}{\mathrm{d}n}\langle 0|\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)|0\rangle\right)_{n=1} \sim \log 2 - \left(\frac{\mathrm{d}}{\mathrm{d}n}\langle 0,+|^{\otimes n}\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)|0,+\rangle^{\otimes n}\right)_{n=1} \tag{4.56}$$

which indeed gives (4.54).

5. Conclusions and discussion

In this paper, we have summarized the fundamentals and main results of a program which was proposed in [31] and developed further over a series of recent publications [30, 32, 52]. This program allows for the computation of the entanglement entropy of a connected region A a massive quantum one-dimensional system with respect to the remaining part of the system. To this aim, it takes full advantage of integrable quantum field theory techniques, as well as analytic scattering theory beyond integrability, by relating the bi-partite entanglement entropy to the correlation functions of a particular kind of local quantum fields: branch-point twist fields.

More precisely, we take as starting point the well-known 'replica trick' and realize that starting with a quantum integrable model and constructing a new model consisting of n non-interacting copies of the original theory, a new local QFT is obtained which naturally possesses

permutation symmetry. Associated with certain elements of this symmetry, two twist fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ exist, whose correlation functions are directly related to the bi-partite entanglement entropy. In general, there are as many branch-point twist fields in a correlation function as there are boundary points of the region A. We considered two cases, where the whole system does not have boundaries (bulk two-point functions), and where the system has one boundary and the region A starts at this boundary (boundary one-point function), see figure 1. Through the replica trick, the entropy is the derivative with respect to n of the correlation function evaluated at $n = 1$.

Since the fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are local fields of the n-copy theory, and the boundary state admits a realization in terms of the scattering states of the bulk model, the correlation functions (two- and one-point functions) above can be computed by using the form factor expansion of massive quantum field theory. They can be expressed as a sum for different particle numbers over products of the form factors of the fields involved. This gives a large-distance expansion: computing the correlation function in the two-particle approximation, for instance, gives the behaviour at large distances. This expansion is in fact expected to converge rapidly, and the two-particle approximation is often enough to describe the correlation function up to relatively small distances.

The leading behaviour of the entropy as a function of the distance was already well known for very short and very large distances, both for bulk and boundary models, before our work and is described by equations (1.9) and (1.11). For $r \ll m^{-1}$ (m being the mass of the lightest particle) that behaviour is fixed by the properties of the underlying CFT which describes the integrable QFT in the ultraviolet limit (in particular, its central charge). Thus, the entropy can be computed explicitly by using CFT techniques [24]. At large separations $r \gg m^{-1}$ (in the infrared limit) the entropy is known to saturate to a constant value.

One of the main results of our work has been to evaluate the first correction to the bulk entropy in the infrared (large-distance) limit, providing therefore a description of the behaviour of the entropy in the intermediate region of values of rm. This correction is obtained from the two-particle contributions to the form factor expansion. In fact, the two-particle approximation provides both the saturation value of the entropy, coming from the disconnected part of the correlation function (the square of the vacuum expectation value of the twist fields), and the exact first corrections up to $O(\mathrm{e}^{-3rm})$, coming from the two-particle contributions. The exact value of the saturation in the Ising model (1.10) was computed in [31] and showed there to be in good agreement with previous numerical results [14].

The most surprising result of this analysis has been to establish that the leading correction to the entropy at large rm is in fact a more universal quantity than expected, that is, it does not depend on the particular scattering matrix of the model we started with, but only on the spectrum of masses of the particles of the original theory. This was shown to be true for all integrable QFTs (with or without backscattering and/or bound states) [30, 31] and even for non-integrable ones [52]. It is quite remarkable that the entropy should encode so explicitly crucial information about the theory both in its UV regime (the central charge) and in the IR regime (the number of light particles).

We have deduced this result from general arguments and checked it explicitly for the Ising, sine-Gordon models and sine-Gordon models. The mathematical reason for this 'universal behaviour' is clear, as the result is directly related to the presence of kinematic poles in the two-particle form factors. Only form factors having such poles do contribute to the final result for the entropy, and their individual contributions turn out to be independent of the scattering matrix. The presence of bound-state poles does not change this conclusion, neither does the loss of integrability.

For the quantum Ising model, we extended the main result above and identified exact infinite-series formulae for the bi-partite entanglement entropy both in the presence and absence of boundaries [32]. In order to obtain our formulae, we found closed expressions for all non-vanishing form factors of branch-point twist fields in the n-copy theory; we identified the correct analytic continuation in n of the contributions of these form factors to correlation functions and evaluated their derivatives with respect to n; and we checked both the form factor formulae and their analytic continuation for consistency. Since Wick's theorem applies, all form factors of the twist field admit expressions in terms of a Pfaffian. From this, obtaining the right analytic continuation of every contribution to the form factor expansion of the twist-field boundary one-point function or bulk two-point function is a highly non-trivial problem. We solved this problem in [32], and verified it for consistency in a very precise manner by finding an explicit formula for the leading logarithmic behaviour both of the two-point function of the twist field in the bulk and of its derivative at $n = 1$, and using a combination of analytical and numerical computations.

The results just described for the Ising model have put us in the position to analyse another quantity of interest in this context, that is the contribution to the free energy that can be attributed exclusively to the presence of the boundary. This is essentially the boundary entropy, the natural logarithm of the boundary degeneracy or g-factor originally introduced by Affleck and Ludwig in [54]. In our analysis, we have computed the universal quantity $V(\kappa)$ which is closely related to g (where κ is related to the boundary magnetic field in the Ising model). We have defined $V(\kappa)$ in a universal, QFT way as a certain rm independent contribution to the boundary entanglement entropy in the UV limit. We have found an exact formula for $V(\kappa)$ and evaluated it at $\kappa = 0, -\infty$ to $V(0) = \log\sqrt{2}$ and $V(-\infty) = 0$. We have also gathered strong numerical evidence that $V(\kappa)$ is in fact constant and equal to $V(0)$ for any finite values of κ. These two values of $V(\kappa)$ would correspond to the two conformal invariant boundary conditions that are known for the Ising model: the free and fixed boundary conditions. The fact that $V(\kappa)$ is larger for free boundary conditions (finite magnetic field) and that $V(0) - V(-\infty) = \log\sqrt{2}$ are properties which also hold for $\log g$. However, it is known from Cardy and Lewellen's work [60] that $g_{\text{free}} = 1$ and $g_{\text{fixed}} = 1/\sqrt{2}$, hence

$$V(\kappa) - \log g = \log\sqrt{2}. \tag{5.1}$$

We have identified the difference between these two quantities as an IR contribution to the entanglement entropy coming from the ground-state degeneracy in the periodic case that is broken in the boundary case. We have proposed a generalization of this to more general models with relevant boundary perturbations.

Although not discussed here in any detail, we have also found interesting results for the IR behaviour of the entanglement entropy in the boundary case. It is known that the bulk entropy saturates for large distances. In particular, the fact that the entropy is an increasing function of rm follows from the 'strong subadditivity theorem' and translation invariance, as proven in [18, 83]. Since translation invariance is broken in the boundary theory, the entanglement entropy in this case is not necessarily a monotonic function of rm. Indeed, we find that, for a range of values of κ, it has a maximum for some value of rm before reaching its asymptotic value. This range of values of κ seems to start at $\kappa = -1$ which corresponds to a value of the magnetic field for which, in a sense, the boundary becomes 'critical'.

There are many open problems related to our work and in general, to the computation of the entanglement entropy in integrable QFT. In the case of the bulk Ising model, it is known that the entanglement entropy can be described via Painlevé transcendents [75, 84]. It would be interesting to check the consistency of this representation with our full form factor expansion. In the boundary Ising case, it is known that the one-point function of the order parameter

has a Fredholm determinant representation for any magnetic field, from which differential equations can be derived [79]. It would be interesting to see if similar formulae hold for the branch-point twist field. In the general QFT case, the most obvious problem is perhaps to extend the present boundary analysis to theories other than the Ising model. We believe that this should be possible to some extent but it is very unlikely that re-summations can be done analytically for interacting models. Yet an independent check of (4.53) in the more general situation would be useful. It would also be interesting to apply the form factor approach to the computation of the entanglement entropy of disconnected regions (see e.g. [85, 86]), both for bulk and boundary theories and also to extend the analysis of the present paper to the finite temperature situation.

An additional result of our work has been to develop the form factor program for branch-point twist fields in integrable models. As a consequence of the particular exchange relations between these fields and the fundamental fields of the theory, the form factor consistency equations for branch-point twist fields are different from those associated with standard local fields. In particular, the crossing and kinematic residue equations are modified. Since here we have only been concerned with the two-particle form factors of branch-point twist fields, an interesting open problem remains, namely to find closed solutions to the form factor equations for arbitrary or at least higher particle numbers for models other than the Ising theory. As explained in section 3.5, we expect these solutions to be given in terms of elementary symmetric polynomials of the variables $e^{\theta_i/n}$. In fact, the form factor program for more general twist fields is an aspect of integrable QFT that still needs much more development.

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